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CONTENTS

THE GENERALIZED WEIERSTRASS APPROXIMATION THEOREM

by

M. H. Stone

1. *Introduction.* Some years ago the writer discovered a generalization of the Weierstrass approximation theorem suggested by an inquiry into certain algebraic properties of the continuous real functions on a topological space [1]. This generalization has since shown itself to be very useful in a variety of similar situations. Interest in it has stimulated several improvements in the proof originally given and has also led to some modifications and extensions of the theorem itself. At the same time many interesting applications to classical problems of analysis have been observed by those working with the generalized approximation theorem. The writer, for instance, has noted a number of such applications in his lectures of 1942-1945, dealing with this and other subjects. Since the proofs thus obtained for several important classical theorems are remarkably simple, there would seem to be some advantage in collecting the relevant material in an expository article where everything could be presented in the light of our most recent knowledge. To offer such an article is our present purpose.

2. *Lattice Formulations of the Generalized Theorem.* The Weierstrass approximation theorem states, of course, that any continuous real function defined on a bounded closed interval of real numbers can be uniformly approximated by polynomials. The generalization with which we shall be concerned here seeks in the first instance to lighten the restrictions imposed on the domain over which the given functions are defined. The difficulty which has to be turned at the very outset in formulating such a generalization is that there are no polynomials on a general domain. It is rather easy, however, to circumvent this difficulty by orienting our inquiry towards the solution of the following question: what functions can be built from the functions of a prescribed family by the application of the algebraic operations (addition, multiplication, and multiplication by real numbers) and of uniform passages to the limit? In the classical case settled by the Weierstrass approximation theorem, the prescribed family consists of just two functions, f_1 and f_2, where $f_1(x) = 1$ and $f_2(x) = x$ for all x in the basic interval. In this, as in other cases which will be noted below, the answer is especially interesting because a very small prescribed family suffices to generate a very much more inclusive family. In his first discussion of the general problem posed above, the author focussed attention on the rôle played in approximation theory by the operations of forming the maximum and the minimum of a pair of functions. The reason why these opera-

tions are technically appropriate to the end in view can be seen even in the classical case of Weierstrass. There it is geometrically evident that a given continuous real function can be uniformly approximated by continuous piecewise linear functions, since to obtain such approximations one has only to inscribe polygons in the graph of the given function; and each piecewise linear function can be obtained from linear functions by means of the operations in question. The approximation of piecewise linear functions by polynomials then becomes the issue. The parts of the author's proof which involve these operations have since been much improved by Kakutani, with the aid of suggestions made by Chevalley, and the results given explicit formulation as a theorem about lattices of continuous functions [2]. Further modifications will be indicated below in the course of our present discussion.

In accordance with the preceding remarks, we shall start with an arbitrary topological space X, the family $\mathfrak{X}$ of all continuous real functions on X, and a prescribed subfamily $\mathfrak{X}_0$ of $\mathfrak{X}$. Our object is to determine the family $\mathfrak{U}(\mathfrak{X}_0)$ of all those functions which can be built from functions in $\mathfrak{X}_0$ by the application of specified algebraic operations and uniform passage to the limit. We shall consider first the case where the specified operations are the lattice operations $\cup$ and $\cap$ defined as follows: $f \cup g = \max(f, g)$ and $f \cap g = \min(f, g)$ are the functions h and k respectively, where $h(x) = \max(f(x), g(x))$ and $k(x) = \min(f(x), g(x))$ for all x in X. Later we shall take up other cases. In general we shall require of X that it be a compact space or even a compact Hausdorff space; but in the course of our preliminary remarks no such restriction will be necessary.

In all the cases we shall consider, $\mathfrak{U}(\mathfrak{X}_0)$ is a part of $\mathfrak{X}$ closed under uniform passage to the limit - in symbols, $\mathfrak{U}(\mathfrak{X}_0) \subset \mathfrak{X}$, $\mathfrak{U}(\mathfrak{U}(\mathfrak{X}_0)) = \mathfrak{U}(\mathfrak{X}_0)$. Let us discuss these statements briefly in the case of the lattice operations. Since $f \cup g$ and $f \cap g$ are continuous whenever f and g are (the mapping of X into the plane given by $x \to (f(x), g(x))$ is continuous and the mappings of the place into the real number system given by $(\xi, \eta) \to \max(\xi, \eta)$ and $(\xi, \eta) \to \min(\xi, \eta)$ respectively are both continuous, so that the composite mappings $x \to \max(f(x), g(x))$ and $x \to \min(f(x), g(x))$ are continuous also) and since the uniform limit of continuous functions is a continuous function, we see that the operations applied in the construction of $\mathfrak{U}(\mathfrak{X}_0)$ work entirely within $\mathfrak{X}$ and hence that $\mathfrak{U}(\mathfrak{X}_0) \subset \mathfrak{X}$. We now observe that $\mathfrak{U}(\mathfrak{X}_0)$ can be constructed in two steps: we first form all the functions obtainable by applying the algebraic operations alone to members of $\mathfrak{X}_0$ and we then form all the functions obtainable from these by uniform passage to the limit. For convenience let us designate the family of functions obtained in the first step by $\mathfrak{U}_1(\mathfrak{X}_0)$ and the family obtained in the second step by $\mathfrak{U}_2(\mathfrak{X}_0)$. It is evident that $\mathfrak{X}_0 \subset \mathfrak{U}_1(\mathfrak{X}_0) \subset \mathfrak{U}_2(\mathfrak{X}_0) \subset \mathfrak{U}(\mathfrak{X})$. We shall show that $\mathfrak{U}_2(\mathfrak{X})$ is closed under the operations allowed and hence that $\mathfrak{U}(\mathfrak{X}) = \mathfrak{U}_2(\mathfrak{X})$. It is then trivial that $\mathfrak{U}(\mathfrak{X})$ is also closed under those operations. It is easy to see that any function f which is a uniform limit of functions f_n in $\mathfrak{U}_2(\mathfrak{X}_0)$ is itself a member of $\mathfrak{U}_2(\mathfrak{X}_0)$: in fact, each f_n can be uniformly approximated by functions in $\mathfrak{U}_1(\mathfrak{X}_0)$ so that, if ϵ is any positive number, f_n and a corresponding function g_n in $\mathfrak{U}_1(\mathfrak{X}_0)$ can be found satisfying the inequalities $|f(x) - f_n(x)| < \epsilon/2$, $|f_n(x) - g_n(x)| < \epsilon/2$, and hence the inequality $|f(x) - g_n(x)| < \epsilon$ for all x in X. It is also fairly easy to see that whenever f and g are in $\mathfrak{U}_2(\mathfrak{X}_0)$ so also are $f \cup g$ and $f \cap g$. For this it is sufficient to observe that, when f and g are uniform limits of the respective sequences f_n and g_n in $\mathfrak{U}_1(\mathfrak{X}_0)$, then $f \cup g$ and $f \cap g$ are uniform limits of the respective sequences $f_n \cup g_n$, and $f_n \cap g_n$ - which are obviously in $\mathfrak{U}_1(\mathfrak{X}_0)$ too. The validity of this observation depends upon the inequalities

$$|\max(\xi,\eta) - \max(\xi',\eta')| \leqq |\xi - \xi'| + |\eta - \eta'|,$$

$$|\min(\xi,\eta) - \min(\xi',\eta')| \leqq |\xi - \xi'| + |\eta - \eta'|,$$

for which formal proofs based on the equations*

$$\max(\xi,\eta) = 1/2(\xi + \eta + |\xi - \eta|)$$

$$\min(\xi,\eta) = 1/2(\xi + \eta + |\xi - \eta|)$$

are easily given. Using these inequalities and choosing n so that $|f(x) - f_n(x)| < \epsilon/2$, $|g(x) - g_n(x)| < \epsilon/2$ for all x in X, we find directly that $|\max(f(x), g(x)) - \max(f_n(x), g_n(x))| < \epsilon$, $|\min(f(x), g(x)) - \min(f_n(x), g_n(x))| < \epsilon$ for x in X. In case we assume X to be compact, every function in X is automatically bounded. By virtue of this assumption, or by virtue of a direct restriction to the bounded continuous functions on X in the general case, we put ourselves in a position to summarize the preceding remarks in a particularly brief form. In fact, if we restrict $\mathfrak{X}$ to consist of the bounded continuous functions on X and define the distance between two bounded functions f and g to be $\sup_{x \in X} |f(x) - g(x)|$, we thereby make $\mathfrak{X}$ into a complete metric space in which metric convergence is equivalent to uniform convergence. The lattice operations are continuous with respect to this metric. As before, when $\mathfrak{X}_0 \subset \mathfrak{X}$ the relations $\mathfrak{X}_0 \subset \mathfrak{U}(\mathfrak{X}_0) \subset \mathfrak{X}$, $\mathfrak{U}(\mathfrak{X}_0) = \mathfrak{U}(\mathfrak{U}(\mathfrak{X}_0))$ are valid. The first states that the uniform limit of bounded continuous functions is a bounded continuous function, the second that $\mathfrak{U}(\mathfrak{X}_0)$ is metrically and algebraically closed. The proof of the latter fact runs as before; but it can be more briefly stated, as follows: if $\mathfrak{U}_1(\mathfrak{X}_0)$ is the family of all "lattice polynomials" formed from $\mathfrak{X}_0$ and $\mathfrak{U}_2(\mathfrak{X}_0)$ is its metric closure, then $\mathfrak{U}_2(\mathfrak{X}_0)$ is obviously metrically closed and the fact that it is algebraically closed with respect to the lattice operations is a simple, direct consequence of their metric continuity.

We are now ready to determine, in the important case where X is compact, what functions belong to $\mathfrak{U}(\mathfrak{X}_0)$.

Theorem 1: *Let X be a compact space, $\mathfrak{X}$ the family of all continuous (necessarily bounded) real functions on X, $\mathfrak{X}_0$ an arbitrary subfamily of $\mathfrak{X}$, and $\mathfrak{U}(\mathfrak{X}_0)$ the family of all functions (necessarily continuous) generated from $\mathfrak{X}_0$ by the lattice operations and uniform passage to the limit. Then a necessary and sufficient condition for a function f in $\mathfrak{X}$ to be in $\mathfrak{U}(\mathfrak{X}_0)$ is that, whatever the points x, y in X and whatever the positive number ϵ, there exist a function f_{xy} obtained by applying the lattice operations alone to $\mathfrak{X}_0$ and such that $|f(x) - f_{xy}(x)| < \epsilon$, $|f(y) - f_{xy}(y)| < \epsilon$.*

Proof: The necessity of the stated condition is trivial. It is the sufficiency which requires discussion. Starting with the functions f_{xy} in $\mathfrak{U}_1(\mathfrak{X}_0)$ we shall construct an approximant for f. Let G_y designate the open set $(z;\ f(z) - f_{xy}(z) < \epsilon)$, where x is fixed. By hypothesis x and y are in G_y, so that the union of all the sets G_y is the entire space X. The compactness of X implies the existence of points $y_1, \ldots, y_n$ such that the union of the sets $G_{y_1}, \ldots, G_{y_n}$ is still the entire space X. Setting $g_x = f_{xy_1} \cup \ldots f_{xy_n} = \max(f_{xy_1}, \ldots, f_{xy_n})$, we see that for any z in X we have $z \in G_{y_k}$ for a suitable choice of k and hence $g_x(z) \geqq f_{xy_k}(z) > f(z) - \epsilon$. On the other hand the fact that $f_{xy}(x) < f(x) + \epsilon$ implies that $g_x(x) < f(x) + \epsilon$. We can now work in a similar manner with the functions g_x. Let H_x designate the open set $(z;\ g_x(z) < f(z) + \epsilon)$. Evidently x is in H_x, so that the union of all the sets H_x is the entire space X. The compactness of X implies the existence of points $x_1, \ldots, x_m$ such that the union of the sets $H_{x_1}, \ldots, H_{x_m}$ is still the entire space X. Setting $h = g_{x_1} \cap \ldots \cap g_{x_m} = \min(g_{x_1}, \ldots, g_{x_m})$, we see that for any z in X we have $z \in H_{x_k}$ for a suitable choice of k and hence $h(z) \leqq g_{x_k}(z) < f(z) + \epsilon$. On the other hand, the fact that $g_x(z) > f(z) - \epsilon$ for all z and all x implies that $h(z) > f(z) - \epsilon$ for all z. Thus we have $|f(z) - h(z)| < \epsilon$ for all z in X. To complete the proof we note that, since only the lattice operations have been used in constructing the functions g_x and h from the functions f_{xy}, these functions are all in $\mathfrak{U}_1(\mathfrak{X}_0)$, as desired.

We may note two simple corollaries, as follows.

Corollary 1: If $\mathfrak{X}_0$ has the property that, whatever the points, x, y, $x \neq y$, in X and whatever the real numbers α and β, there exists a function f_0 in $\mathfrak{X}_0$ for which $f_0(x) = \alpha$ and $f_0(y) = \beta$, then $\mathfrak{U}(\mathfrak{X}_0) = \mathfrak{X}$ — in other words, any continuous function on X can be uniformly approximated by lattice polynomials in functions belonging to the prescribed family $\mathfrak{X}_0$.

Corollary 2: If a continuous real function on a compact space X is the limit of a monotonic sequence f_n of continuous functions, then the sequence converges uniformly to f.†

† Professor Andre Weil remarks that the extension to monotonic sets is immediate.

Proof: We take $\mathfrak{X}_0$ to be the totality of functions occurring in the sequence f_n. Then $\mathfrak{U}_1(\mathfrak{X}_0) = \mathfrak{X}_0$ since monotonicity implies that $f_m \cup f_n$ coincides with one of the two functions f_m and f_n while $f_m \cap f_n$ coincides with the other. The assumption that $\lim_{n\to\infty} f_n(x) = f(x)$ for every x now shows that the condition of Theorem 1 is satisfied. Hence f is in $\mathfrak{U}(\mathfrak{X}_0)$; and f is therefore the uniform limit of functions occurring in $\mathfrak{X}_0$. Since $|f(x) - f_n(x)|$ decreases as n increases and since $|f(x) - f_N(x)| < \epsilon$ for all x and a suitable choice of N, we see that $|f(x) - f_n(x)| < \epsilon$ for all $n \geq N$, as was to be proved.

Theorem 1 tells us that the question, "Can a given function f be approximated in terms of the prescribed family $\mathfrak{X}_0$?" has an answer depending only on the way in which f and $\mathfrak{X}_0$ behave on pairs of points in X. The contraction of a function obtained by suppressing all points of X except the two points x, y of a pair is a function of very simple kind - it is completely described by the ordered pair (α,β) of those real numbers which are its values at x and at y respectively. If $\mathfrak{X}_0(x,y)$ designates the family of functions obtained by contracting every function in $\mathfrak{X}_0$ in this manner, and if $\mathfrak{X}(x,y)$ has a corresponding significance, then everything depends on an examination (for all different pairs x,y) of the question, "Can a given element of $\mathfrak{X}(x,y)$ be approximated in terms of $\mathfrak{X}_0(x,y)$?" This question is that special case of our original problem in which X is a two-element space! When X has just two elements, the approximation problem can be described in slightly different language, as follows. We have to deal with all ordered pairs (α,β) of real numbers - that is, with the cartesian plane. On two such pairs we can perform the operations $\cup$ and $\cap$ defined by the equations

$$(\alpha,\beta)\cup(\gamma,\delta) = (\max(\alpha,\gamma), \max(\beta,\delta))$$

$$(\alpha,\beta)\cap(\gamma,\delta) = (\min(\alpha,\gamma), \min(\beta,\delta)).$$

Geometrically these operations produce the upper right vertex and lower left vertex respectively of a rectangle with its sides parallel to the coördinate axes and one pair of opposite vertices falling on the points (α,β), (γ,δ). For any given subset S of the plane the problem to be solved is that of finding what points can be generated from it by the above operations and passage to the limit. From what has been said above, it is clear that the points so generated constitute a closed subset S^* of the plane which contains with (α,β) and (γ,δ) the two points described above. It is also clear that this subset is the smallest set enjoying these properties and containing the given subset S. Reverting now to the interpretation of Theorem 1, we see that it can be restated in the following form: if $f \in \mathfrak{X}$, then $f \in \mathfrak{U}(\mathfrak{X}_0)$ if and only if $(f(x), f(y)) \in \mathfrak{X}_0(x,y)^*$ for every pair of distinct points

x, y in X. We have not asserted that the conditions corresponding to various pairs x, y are independent of one another, Nor have we asserted that every point (α,β) in $\mathfrak{X}_0(x,y)^*$ can be expressed in the form $\alpha = f(x)$, $\beta = f(y)$ for some f in $\mathfrak{U}(\mathfrak{X}_0)$. Indeed, even in the case where $\mathfrak{X}_0 = \mathfrak{U}(\mathfrak{X}_0)$ we know only that $\mathfrak{X}_0(x,y)^*$ is the closure of $\mathfrak{X}_0(x,y)$.

It is convenient to express some of the results sketched in the preceding paragraph as a formal theorem. This we do as follows.

Theorem 2: *Let X be a compact space, $\mathfrak{X}$ the family of continuous real functions on X, and $\mathfrak{X}_0$ a subfamily of $\mathfrak{X}$ which is closed under the lattice operations and uniform passage to the limit. Then $\mathfrak{X}_0$ is completely characterized by the system of planar sets $\mathfrak{X}_0(x,y)^* = \mathfrak{X}_0(x,y)^-$.*

Proof: Our hypothesis that $\mathfrak{X}_0 = \mathfrak{U}(\mathfrak{X}_0)$ shows that $\mathfrak{X}_0(x,y)$ has $\mathfrak{X}_0(x,y)^*$ as its closure, as we remarked above. Let us suppose that $\mathfrak{Y}_0 = \mathfrak{U}(\mathfrak{Y}_0)\subset\mathfrak{X}$ and that $\mathfrak{X}_0(x,y)^* = \mathfrak{Y}_0(x,y)^*$ for all pairs of points x, y in X. Then the conditions for f in $\mathfrak{X}$ to belong to $\mathfrak{X}_0$ are identical to those for it to belong to $\mathfrak{Y}_0$. Hence $\mathfrak{X}_0$ and $\mathfrak{Y}_0$ coincide.

We pass now to the modifications of Theorems 1 and 2 which result when we take into consideration the operations of linear algebra as well as the lattice operations. The newly admitted operations are, more precisely, addition and multiplication by real numbers. In view of the equations (*), which express the lattice operations in terms of the linear operations and the single operation of forming the absolute value, we may take the specified algebraic operations to be simply addition, multiplication by real numbers, and formation of absolute values. The remarks preliminary to Theorem 1 apply, mutatis mutandis, to the present situation. The family $\mathfrak{U}(\mathfrak{X}_0)$ of all functions which can be constructed from $\mathfrak{X}_0\subset\mathfrak{X}$ by application of the linear lattice operations and uniform passage to the limit is again seen to be obtainable in two steps, the first being algebraic and the second consisting in the adjunction of uniform limits. This family is closed under the operations used to generate it. We now have the following analogue of the results contained in Theorems 1 and 2.

Theorem 3: *(Kakutani [2]. Let X be a compact space, $\mathfrak{X}$ the family of all continuous (necessarily bounded) real functions on X, $\mathfrak{X}_0$ an arbitrary subfamily of $\mathfrak{X}$, and $\mathfrak{U}(\mathfrak{X}_0)$ the family of all functions (necessarily continuous) generated from $\mathfrak{X}_0$ by the linear lattice operations and uniform passage to the limit. Then a necessary and sufficient condition for a function f in $\mathfrak{X}$ to be in $\mathfrak{U}(\mathfrak{X}_0)$ is that f satisfy every linear relation of the form $\alpha g(x) = \beta g(y)$, $\alpha\beta \geq 0$, which is satisfied by all functions in $\mathfrak{X}_0$. If $\mathfrak{X}_0$ is a closed linear sublattice of $\mathfrak{X}$ - that is, if $\mathfrak{X}_0 = \mathfrak{U}(\mathfrak{X}_0)$ - then $\mathfrak{X}_0$ is characterized by the system of all the linear relations of this form which are satisfied by every function belonging to it. The linear relations associated with an arbitrary pair of points x,y in X must be equivalent to one of the following distinct types:*

(1) *$g(x) = 0$ and $g(y) = 0$;*
(2) *$g(x) = 0$ and $g(y)$ unrestricted, or vice versa;*
(3) *$g(x) = g(y)$ without restriction on the common value;*
(4) *$g(x) = \lambda g(y)$ or $g(y) = \lambda g(x)$ for a unique value λ, $0 < \lambda < 1$.*

Proof: Since $\mathfrak{Y}_0 = \mathfrak{U}(\mathfrak{X}_0)$ is closed under the lattice operations and uniform passage to the limit, Theorem 2 can be applied to $\mathfrak{Y}_0$. However, the fact that $\mathfrak{Y}_0$ is also closed under the linear operations can be expected to produce effective simplifications. Indeed we see that the planar set $\mathfrak{Y}_0(x,y)$, where x and y are arbitrary points in X, must be the entire plane, a straight line passing through the origin, or the one-point set consisting of the origin alone. This appears at once when we observe that if $(\alpha,\beta) \in \mathfrak{Y}_0(x,y)$ then $(\lambda\alpha,\lambda\beta) \in \mathfrak{Y}_0(x,y)$ for every λ, and that if (α,β) and (γ,δ) are in $\mathfrak{Y}_0(x,y)$ then $(\alpha+\gamma,\beta+\delta) \in \mathfrak{Y}_0(x,y)$. Since $\mathfrak{Y}_0(x,y)$ is obviously a closed subset of the plane, we have $\mathfrak{Y}_0(x,y)^* = \mathfrak{Y}_0(x,y)$. When $\mathfrak{Y}_0(x,y)$ is a straight line through the origin we write its equation as $\alpha\xi = \beta\eta$ and observe that $(\beta,\sigma) \in \mathfrak{Y}_0(x,y)$. Since $\mathfrak{Y}_0$ is closed under the operation of forming absolute values, we see that $(|\beta|, |\alpha|) \in \mathfrak{Y}_0(x,y)$. Hence $\alpha|\beta| = |\alpha|\beta$ so that $\alpha\beta|\beta| = |\alpha|\beta^2 \geq 0$ and $\alpha\beta \geq 0$. When $\mathfrak{Y}_0(x,y)$ consists of the origin alone, we have the case enumerated as (1) in the statement of the theorem. When $\mathfrak{Y}_0(x,y)$ is a straight line through the origin we have case (2) if it coincides with one of the coördinate axes, case (3) if it coincides with the bisector of the angle between the positive coördinate axes, and case (4) otherwise. When $\mathfrak{Y}_0(x,y)$ is the entire plane there is no corresponding linear relation, of course. Theorem 2 shows that $\mathfrak{Y}_0$ is characterized by the sets $\mathfrak{Y}_0(x,y) = \mathfrak{Y}_0(x,y)^*$ - in other words, that f in $\mathfrak{X}$ belongs to $\mathfrak{Y}_0 = \mathfrak{U}(\mathfrak{X}_0)$ if and only if $(f(x), f(y)) \in \mathfrak{Y}_0(x,y)$. Since $\mathfrak{X}_0 \subset \mathfrak{Y}_0$, it is clear that the conditions thus imposed on the functions in $\mathfrak{U}(\mathfrak{X}_0)$ are satisfied by the functions in $\mathfrak{X}_0$. On the other hand if all the functions in $\mathfrak{X}_0$ satisfy relations of the kind enumerated in (1) - (4) it is clear that every function in $\mathfrak{U}(\mathfrak{X}_0)$ must do likewise: for the sums, constant multiples, absolute values, and uniform limits of functions which satisfy a condition of any one of these types must satisfy the same condition. Thus the linear relations of the form $\alpha g(x) = \beta g(y)$, $\alpha\beta \geq 0$, satisfied by the functions in $\mathfrak{X}_0$ are identical with those satisfied by the functions in $\mathfrak{U}(\mathfrak{X}_0)$ and serve to characterize the latter family completely.

We may note some simple corollaries to the theorem just proved.

Corollary 1: *In order that $\mathfrak{U}(\mathfrak{X}_0)$ contain a non-vanishing constant function it is necessary and sufficient that the only linear relations of the form $\alpha g(x) = \beta g(y)$, $\alpha\beta > 0$, satisfied by every function in $\mathfrak{X}_0$ be those reducible to the form $g(x) = g(y)$.*

Proof: It is obvious that of conditions (1) - (4) in Theorem 3 only condition (3) can be satisfied by a non-vanishing constant function.

Corollary 2: In order that $\mathfrak{U}(\mathfrak{X}_0) = \mathfrak{X}$ *it is sufficient that the functions in* $\mathfrak{X}_0$ *satisfy no linear relation of the form* (1) - (4) *of Theorem 3.*

In order to state a further corollary, we first introduce a convenient definition.

Definition 1: A family of arbitrary functions on a domain X *is said to be a separating family (for that domain) if, whenever* x *and* y *are distinct points of* X, *there is some function* f *in the family with distinct values* $f(x)$, $f(y)$ *at these points.*

In terms of this definition we have the following result.

Corollary 3: *If* X *is compact and if* $\mathfrak{X}_0$ *is a separating family for* X *and contains a non-vanishing constant function, then* $\mathfrak{U}(\mathfrak{X}_0) = \mathfrak{X}$.

Proof: Since $\mathfrak{X}_0$ contains a non-vanishing constant function, the only one of conditions (1) - (4) satisfied by every function in $\mathfrak{X}_0$ are those of the form (3). Since $\mathfrak{X}_0$ is a separating family, no linear relation of the form $g(x) = g(y)$, where $x \neq y$, is satisfied by every function in $\mathfrak{X}_0$. Hence Corollary 2 yields the desired result.

Corollary 4: *If* $\mathfrak{X}_0$ *is a separating family, then so is* $\mathfrak{X}$. *If* $\mathfrak{X}$ *is a separating family and* $\mathfrak{U}(\mathfrak{X}_0) = \mathfrak{X}$, *then* $\mathfrak{X}_0$ *is also a separating family.*

Proof: The first statement is trivial. The second statement follows at once from the fact that $\mathfrak{X}_0$ is subject to no linear relation of the form $g(x) = g(y)$ which is not also satisfied by every function in $\mathfrak{U}(\mathfrak{X}_0) = \mathfrak{X}$.

It should be remarked that in general the family $\mathfrak{X}$ of all continuous functions on a compact space X need not be a separating family. In case X is a compact Hausdorff space, however, it is well-known that $\mathfrak{X}$ is a separating family: if $x \neq y$, there exists a continuous function f on X such that $f(x) = 0$, $f(y) = 1$.

3. *Linear Ring Formulations of the Generalized Theorem.* We are now ready to discuss the approximation problem when the specified algebraic operations used in the construction of approximants are the linear operations and multiplication. Since the product of two continuous functions is continuous we see that the family $\mathfrak{X}$ of all continuous functions on a topological space $\mathfrak{X}$ is a commutative ring with respect to the two operations of addition and multiplication, and a commutative linear associative algebra or linear ring with respect to the operations of addition, multiplication, and multiplication by real numbers. Hence the formally stated results of this section constitute what may be called the linear-ring formulation of the generalized Weierstrass approximation theorem.

If we now designate by $\mathfrak{U}(\mathfrak{X}_0)$ the family of all functions generated from $\mathfrak{X}_0 \subset \mathfrak{X}$ by means of the linear-ring operations and uniform passage of the limit, we have to note a slight modification which must be made in the general statements made in the lattice case. If f and g are uniform limits of the sequences f_n and g_n respectively, the product fg is not in general the uniform limit of the sequence $f_n g_n$ - consider, for example, the case where $f = g$ is a non-bounded function and $f - f_n$ is the constant $1/n$. *We shall therefore suppose that $\mathfrak{X}$ consists of all bounded continuous functions on a topological space X,* this boundedness restriction being automatically satisfied when X is compact. By virtue of this restriction we can apply the inequality

$$|fg - f_n g_n| \leq |f||g - g_n| + |g||f - f_n| + |f - f_n||g - g_n|$$

to show that when f_n and g_n are uniformly convergent sequences in $\mathfrak{X}$ their respective limits f and g are in $\mathfrak{X}$ and that the sequence $f_n g_n$ converges uniformly to the product fg, in $\mathfrak{X}$. When $\mathfrak{X}_0 \subset \mathfrak{X}$ we see as before that $\mathfrak{U}(\mathfrak{X}_0) \subset \mathfrak{X}$, $\mathfrak{U}(\mathfrak{U}(\mathfrak{X}_0)) = \mathfrak{U}(\mathfrak{X}_0)$. It is easy to see that $\mathfrak{U}(\mathfrak{X}_0)$ consists of all those functions, necessarily in $\mathfrak{X}$, which are uniform limits of polynomials in members of $\mathfrak{X}_0$ - in other words, $f \in \mathfrak{X}$ is in $\mathfrak{U}(\mathfrak{X}_0)$ if and only if, whatever the positive number ϵ, there exist functions $f_1, \ldots, f_n$ and a polynomial function $p(\xi_1, \ldots, \xi_n)$ of the real variables $\xi_1, \ldots, \xi_n$ with $p(0, \ldots, 0) = 0$ such that $|f(x) - p(f_1(x), \ldots, f_n(x))| < \epsilon$ for every x in X.

Now in order to prove our principal theorem we shall establish a very special case of the classical Weierstrass approximation theorem, using for this purpose direct and elementary methods which do not depend on any general theory. The result we need is the following proposition.

Theorem 4: *If ϵ is any positive number and $\alpha \leq \xi \leq \beta$ any real interval, then there exists a polynomial $p(\xi)$ in the real variable ξ with $p(0) = 0$ such that $||\xi| - p(\xi)| < \epsilon$ for $\alpha \leq \xi \leq \beta$.*

Proof: Unless the point $\xi = 0$ is inside the given interval (α, β), we can obviously take $p(\xi) = \pm\,\xi$. Thus there is no loss of generality in confining our attention to intervals of the form $(-\gamma, \gamma)$ where $\gamma > 0$, since the given interval (α, β) can be included in an interval of this form. Moreover it is obviously sufficient to study the case of the interval $(-1, 1)$ since, if $q(\eta)$, $q(0) = 0$, is a polynomial such that $||\eta| - q(\eta)| < \epsilon/\gamma$ for $-1 \leq \eta \leq 1$, then $p(\xi) = \gamma q(\xi/\gamma)$, $p(0) = 0$, is a polynomial such that $||\xi| - p(\xi)| < \epsilon$ for $-\gamma \leq \xi \leq \gamma$. We shall obtain the desired polynomial q for the interval $-1 \leq \eta \leq 1$ as a partial sum of the power series development for $\sqrt{1 - \zeta}$ where $\zeta = 1 - \eta^2$. The validity of the development has to be established directly.

We commence by defining a sequence of constants α_k recursively from the relations

$$\alpha_1 = 1/2, \; \alpha_k = \frac{1}{2} \sum_{m, n \geq 1}^{m+n=k} \alpha_m \alpha_n \text{ for } k \geq 2.$$

It is obvious that $\alpha_k > 0$. Putting $\sigma_n = \sum_{k=1}^{k=n} \alpha_k$ we can show inductively that $\sigma_n < 1$. In fact we have $\sigma_1 = \alpha_1 = 1/2 < 1$ and note that $\sigma_n < 1$ implies

$$\sigma_{n+1} = \alpha_1 + \sum_{k=2}^{n+1} \alpha_k = \frac{1}{2} + \frac{1}{2} \sum_{k=2}^{n+1} \sum_{i,j\geq 1}^{i+j=k} \alpha_i \alpha_j \leq \frac{1}{2} + \frac{1}{2} \sum_{i,j=1}^{n} \alpha_i \alpha_j$$

$$\leq \frac{1}{2}(1 + \alpha_n^2) < 1.$$

Accordingly the positive term series $\sum_{k=1}^{\infty} \alpha_k$ converges to a sum σ satisfying the inequality $\sigma \leq 1$; and the power series $\sum_{k=1}^{\infty} \alpha_k \zeta^k$ converges uniformly for $|\zeta| \leq 1$ to a continuous function $\sigma(\zeta)$. It is now comparatively easy to identify this function with the function $1 - 1\sqrt{-\zeta}$. To do so we prove that $\sigma(\zeta)(2 - \sigma(\zeta)) = \zeta$. Looking at the partial sums of the power series for $\sigma(\zeta)$, we observe that

$$\left(\sum_{i=1}^{n} \alpha_i \zeta^i\right)\left(2 - \sum_{j=1}^{n} \alpha_j \zeta^j\right) = 2\sum_{k=1}^{n} \alpha_k \zeta^k - \sum_{i,j=1}^{n} \alpha_i \alpha_j \zeta^{i+j}$$

$$= 2\sum_{k=1}^{n} \alpha_k \zeta^k - 2\sum_{k=2}^{n} \alpha_k \zeta^k - \sum_{1\leq i,j\leq n}^{i+j\geq n+1} \alpha_i \alpha_j \zeta^{i+j}$$

$$= \zeta - \sum_{1\leq i,j\leq n}^{i+j\geq n+1} \alpha_i \alpha_j \zeta^{i+j}$$

in accordance with the definition of the coefficients α_k. The final term here can now be estimated as follows:

$$\Big| \sum_{1\le i,\, j\le n}^{i+j\ge n+1} \alpha_i \alpha_j \zeta^{i+j} \Big| \le \sum_{1\le i,\, j\le n}^{i+j\ge n+1} \alpha_i\alpha_j \le \sum_{k=n+1}^{\infty} \sum_{i,\, j\ge 1}^{i+j=k} \alpha_i\alpha_j$$

$$\le 2 \sum_{k=n+1}^{\infty} \alpha_k .$$

When n becomes infinite, therefore, this term tends to zero; and passage to the limit in the identity above accordingly yields the relation $\sigma(\zeta)(2 - \sigma(\zeta)) = \zeta$. For each ζ such that $-1 \le \zeta \le 1$ we have $\sigma(\zeta) = 1 \pm \sqrt{1 - \zeta}$. Here we decide upon the choice of sign by showing that $\sigma(\zeta) \le 1$, an inequality incompatible with the upper sign. It is evident that $\sigma(1) = 1$, independently of the choice of sign, and hence that $\sum_{k=1}^{\infty} \alpha_k = \sigma(1) = 1$. Inasmuch as α_k is positive it follows that $\sigma(\zeta) \le \sigma(|\zeta|) \le \sigma(1) = 1$, as we intended to show. It is now clear that the power series for $\sqrt{1 - \zeta}$ is given by

$$\sqrt{1 - \zeta} = 1 - \sigma(\zeta) = 1 - \sum_{k=1}^{\infty} \alpha_k \zeta^k = \sum_{k=1}^{\infty} \alpha_k (1 - \zeta^k).$$

Taking η so that $-1 \le \eta \le 1$, we have $0 \le 1 - \eta^2 \le 1$ and hence

$$|\eta| = \sqrt{\eta^2} = 1 - \sigma(1 - \eta^2) = \sum_{k=1}^{\infty} \alpha_k (1 - (1 - \eta^2)^k),$$

the series being uniformly convergent. The general term of this series is a polynomial in η which vanishes for $\eta = 0$. Hence we can take a suitable one of its partial sums as the required polynomial $q(\eta)$, thus completing our discussion.

We are now ready to give our principal results concerning the generalization of the Weierstrass theorem for the linear-ring operations.

Theorem 5: *Let X be a compact space, $\mathfrak{X}$ the family of all continuous real functions on X, $\mathfrak{X}_0$ an arbitrary subfamily of $\mathfrak{X}$, and $\mathfrak{U}(\mathfrak{X}_0)$ the family of all functions (necessarily continuous) generated from $\mathfrak{X}_0$ by the linear ring operations and uniform passage to* *the limit. Then a necessary and sufficient condition for a function f in $\mathfrak{X}$ to be in $\mathfrak{U}(\mathfrak{X}_0)$ is that f satisfy every linear relation of the form $g(x) = 0$ or $g(x) = g(y)$ which is satisfied by all functions in $\mathfrak{X}_0$. If $\mathfrak{X}_0$ is a closed linear subring of $\mathfrak{X}$ - that is, if $\mathfrak{X}_0 = \mathfrak{U}(\mathfrak{X}_0)$ - then $\mathfrak{X}_0$ is characterized by the system of all the linear relations of this kind which are satisfied by every function belonging to it. In other words, $\mathfrak{X}_0$ is characterized by the partition of X into mutually disjoint closed subsets on each of which every function in $\mathfrak{X}_0$ is constant and by the specification of that one, if any, of these subsets on which every function in $\mathfrak{X}_0$ vanishes.*

Proof: By virtue of Theorem 4, we see that if f is in $\mathfrak{U}(\mathfrak{X}_0)$ then $|f|$ is also in $\mathfrak{U}(\mathfrak{X}_0)$. Indeed, since X is compact, the function f is bounded. Assuming accordingly that $\alpha \leq f(x) \leq \beta$ for all x, we can find a polynomial $p_n(\xi)$ such that $||\xi| - p_n(\xi)| < 1/n$ for $\alpha \leq \xi \leq \beta$, while $p_n(0) = 0$. It is clear that $p_n(f)$ is in $\mathfrak{U}(\mathfrak{X}_0)$ and that $||f(x)| - p_n(f(x))| < 1/n$ for all x in X. Hence $|f|$ is the uniform limit of functions - namely, the functions $p_n(f)$ - in $\mathfrak{U}(\mathfrak{X}_0)$. Thus $|f|$ is in $\mathfrak{U}(\mathfrak{X}_0)$, as we wished to prove. By virtue of the formulas (*) connecting the operations $\cup$ and $\cap$ with the operation of forming the absolute value, we now see that whenever f and g are in $\mathfrak{U}(\mathfrak{X}_0)$ then so also are $f \cup g$ and $f \cap g$ - in other words, $\mathfrak{U}(\mathfrak{X}_0)$ is closed under the linear lattice operations, as well as under the ring operations and uniform passage to the limit. The characterization of closed linear sublattices of $\mathfrak{X}$ given in Theorem 3 applies, naturally, to $\mathfrak{U}(\mathfrak{X}_0)$. It is easy to see that none of the characteristic linear relations can be of the type (4) described there. In fact, if every function in $\mathfrak{U}(\mathfrak{X}_0)$ were to satisfy a linear relation of the form $g(x) = \lambda g(y)$, we would find for every f in $\mathfrak{U}(\mathfrak{X}_0)$ that, f^2 being also in $\mathfrak{U}(\mathfrak{X}_0)$, the relations $f(x) = \lambda f(y)$ $f^2(x) = \lambda f^2(y)$, $\lambda^2 f^2(y) = \lambda f^2(y)$ would hold; and we would conclude that $f(y) = 0$ for every f in $\mathfrak{U}(\mathfrak{X}_0)$ or that $\lambda = 0, 1$. Thus we conclude that f is in $\mathfrak{U}(\mathfrak{X}_0)$ if and only if it satisfies all the linear relations $g(x) = 0$ or $g(x) = g(y)$ satisfied by every function in $\mathfrak{X}_0$. The first characterization of the closed linear subrings of $\mathfrak{X}$ given in the statement of the theorem follows immediately. As to the second characterization, we remark first that the relation $\equiv$ defined by putting $x \equiv y$ if and only if $f(x) = f(y)$ for all f in $\mathfrak{X}_0$ is obviously an equivalence relation *stronger* than the natural equality in X: $x = y$ implies $x \equiv y$; $x \equiv y$ implies $y \equiv x$; $x \equiv y$ and $y \equiv z$ imply $x \equiv z$. Consequently, X is partitioned by this equivalence-relation into mutually disjoint subsets, each a maximal set of mutually equivalent elements. The set of all points y such that $x \equiv y$ is just that partition-class which contains x. Since this set is the intersection or common part of all the sets $X_f = (y;\ f(x) = f(y))$ for the various functions f in $\mathfrak{X}_0$ and since each set X_f is closed by virtue of the continuity of f, we see that the partition-class containing x is closed. If x and y are in distinct partition-classes, then there exists a function f in $\mathfrak{X}_0$ such that $f(x) \neq f(y)$, since otherwise we would have $x \equiv y$ and the two given partition-classes could not be distinct. If a partition-class contains a single point x such that $f(x) = 0$ for every f in $\mathfrak{X}_0$, then all its points obviously have this property. On the other hand, at most one partition-class can contain such a point since, if x and y are points such that $f(x) = 0$, $f(y) = 0$ for every f in $\mathfrak{X}_0$, then $f(x) = f(y)$ for every f in $\mathfrak{X}_0$, $x \equiv y$, and x and y are in the same partition-class. We cannot expect that an arbitrary partition of X into mutually disjoint closed subsets can be derived in the manner just described from some closed linear subring $\mathfrak{X}_0$ of $\mathfrak{X}$. However, partitions obtained from *distinct* closed linear subrings are necessarily *distinct* - except in the case where one subring consists of all the functions in $\mathfrak{X}$ which are constant on each partition-class and the other consists of all those functions which are in the first subring and in addition vanish on one specified partition-class. Thus we see that a closed linear subring is specified by the partition of X into the closed subsets on each of which all its members are constant and the specification of that particular partition-class, if any, on which all its members vanish.

We have at once a pair of useful corollaries.

Corollary 1: *In order that $\mathfrak{U}(\mathfrak{X}_0)$ contain a non-vanishing constant function it is necessary and sufficient that for every x in X there exist some f in $\mathfrak{X}_0$ such that $f(x) \neq 0$.*

Corollary 2: *If $\mathfrak{X}_0$ is a separating family for X, then $\mathfrak{U}(\mathfrak{X}_0)$ either coincides with $\mathfrak{X}$ or is, for a uniquely determined point x_0, the family of all functions f in $\mathfrak{X}$ such that $f(x_0) = 0$. If, conversely, $\mathfrak{X}$ is a separating family for X and $\mathfrak{U}(\mathfrak{X}_0)$ either coincides with $\mathfrak{X}$ or is the family of all those f in $\mathfrak{X}$ which vanish at some fixed point x_0 in X, then $\mathfrak{X}_0$ is a separating family.*

Proof: If $\mathfrak{X}_0$ is a separating family, so also are $\mathfrak{U}(\mathfrak{X}_0)$ and $\mathfrak{X}$. Hence the partition-classes associated with $\mathfrak{U}(\mathfrak{X}_0)$ must each consist of a single point. It follows that $\mathfrak{U}(\mathfrak{X}_0)$ must be as indicated. Conversely, when $\mathfrak{X}$ is a separating family and $\mathfrak{U}(\mathfrak{X}_0)$ is as stated, then $\mathfrak{U}(\mathfrak{X}_0)$ is a separating family. If it were not every f in $\mathfrak{U}(\mathfrak{X}_0)$ vanishes at some point x_0; and there would exist distinct points x and y in X such that $f_0(x) = f_0(y)$ for every f_0 in $\mathfrak{U}(\mathfrak{X}_0)$. Consider now an arbitrary function f in $\mathfrak{X}$. Clearly the function f_0 defined by putting $f_0(z) = f(z) - f(x_0)$ is continuous and vanishes at x_0. Thus f_0 is in $\mathfrak{U}(\mathfrak{X}_0)$, the equation $f_0(x) = f_0(y)$ is verified, and in consequence $f(x) = f(y)$. Thus we find that $f(x) = f(y)$ for every f in $\mathfrak{X}$, against hypothesis. Since $\mathfrak{U}(\mathfrak{X}_0)$ is a separating family, $\mathfrak{X}_0$ must be also. Otherwise, of course, there would exist distinct points x, y in X such that $f_0(x) = f_0(y)$ for every f_0 in $\mathfrak{X}_0$; and then the equation $f(x) = f(y)$ would hold for every f in $\mathfrak{U}(\mathfrak{X}_0)$, contrary to what was just established.

4. *The Characterization of Closed Ideals.* In developing effective general algebraic theories of lattices, linear lattices, and rings, it has been found useful to introduce the concept of an

ideal. Although ideals are independently defined in the different algebraic circumstances mentioned, their theoretical rôles do not differ much from one case to another. Because the results of the preceding sections easily yield characterizations of those ideals in $\mathfrak{X}$ (the family of all continuous real functions on a compact space X) which are closed under uniform passage to the limit, it seems worthwhile to digress from the main line of our discussion long enough to present the very useful facts available in this domain. This we shall now do without further detailed analysis of the concept of an ideal.

When we think of $\mathfrak{X}$ as a lattice - the only algebraic operations taken into consideration being the operations $\cup$ and $\cap$ - we define† an ideal $\mathfrak{X}_0$ to be a non-void subclass of $\mathfrak{X}$ which contains $f \cup g$ together with f and g, and which contains $f \cap g$ together with f. The second condition of this definition is evidently equivalent to the requirement that $\mathfrak{X}_0$ should contain g whenever it contains f and $f(x) \geq g(x)$ for every x. We now have the following characterization of the closed ideals in $\mathfrak{X}$.

Theorem 6: *Let $\mathfrak{X}$ be the lattice of all continuous real functions on a compact space X, $\mathfrak{X}_0$ an arbitrary subfamily of $\mathfrak{X}$, F_0 the extended-real function defined on X through the equation $F_0(x) = \sup_{f \in \mathfrak{X}_0} f(x)$, and $\mathfrak{Y}_0$ the family of all those functions f in $\mathfrak{X}$ such that $f(x) \leq F_0(x)$ for every x in X. When $\mathfrak{X}_0$ is void, $F_0(x) = -\infty$ for every x and $\mathfrak{Y}_0$ is void. Otherwise, $\mathfrak{Y}_0$ is the smallest closed ideal containing $\mathfrak{X}_0$; and $\mathfrak{X}_0$ is a closed ideal if and only if $\mathfrak{X}_0 = \mathfrak{Y}_0$. A closed ideal $\mathfrak{X}_0$ is characterized by the associated function F_0.*

Proof: As indicated in the statement of the theorem, we permit $+\infty$ and $-\infty$ to appear as values of F_0, when necessary. When $\mathfrak{X}_0$ is non-void, it is easy to verify that $F_0(x) > -\infty$ for every x and that $\mathfrak{Y}_0$ is non-void and is a closed ideal in $\mathfrak{X}$. For example, if f is in $\mathfrak{Y}_0$ and $g(x) \leq f(x)$ for every x, then obviously $g(x) \leq F_0(x)$ for every x and g is in $\mathfrak{Y}_0$. If $\mathfrak{X}_0$ is a closed ideal, we can show that $\mathfrak{X}_0 = \mathfrak{Y}_0$. To do so we examine the relations between F_0 and the planar sets $\mathfrak{X}_0(x,y)^*$ which characterize $\mathfrak{X}_0$ as a closed sublattice of $\mathfrak{X}$ in accordance with Theorem 2. First of all, it is evident that $\mathfrak{X}_0(x,y)$, and hence also its closure $\mathfrak{X}_0(x,y)^*$, must be contained in the set of points (α,β) such that $\alpha \leq F_0(x)$ and $\beta \leq F_0(y)$. On the other hand, $(F_0(x), F_0(y))$ is a limit point of $\mathfrak{X}_0(x,y)$ and is therefore in $\mathfrak{X}_0(x,y)^*$, as will be verified at once. If $\epsilon < 0$, then there exist functions f and g in $\mathfrak{X}_0$ such that $f(x) > F_0(x) - \epsilon$, $g(y) > F_0(y) - \epsilon$ for any prescribed pair of points x, y in X. The function $h = f \cup g$ is in the ideal $\mathfrak{X}_0$ and satisfies the

†Because of the familiar duality between the two operation $\cup$ and $\cap$, there is also a dual definition in which the roles played here by these operations are interchanged.

relations $h(x) \geq f(x) > F\ (x) - \epsilon$, $h(y) \geq g(y) > F_0(y) - \epsilon$. Thus $(h(x), h(y))$ is a point in $\mathfrak{X}_0(x,y)$ and $|h(x) - F_0(x)| < \epsilon$, $|h(y) - F_0(y)| < \epsilon$, so that $(F_0(x), F_0(y))$ is in $\mathfrak{X}_0(x,y)^*$ as we wished to prove. Now we establish the fact that f is in $\mathfrak{X}_0$ when $f(x) \leq F_0(x)$ for every x. Let f_ϵ be the function in $\mathfrak{X}$ defined by putting $f_\epsilon(x) = f(x) - \epsilon$, $\epsilon > 0$. If x, y are arbitrary points in X, an argument similar to that just carried through shows that there exists a function h in $\mathfrak{X}_0$ satisfying the inequalities $h(x) > f_\epsilon(x) = f(x) - \epsilon$, $h(y) > f_\epsilon(y) = f(y) - \epsilon$. The function $f_{xy} = h \cap f_\epsilon$ then belongs to the ideal $\mathfrak{X}_0$ and has the property that $f(x) - f_{xy}(x) = \epsilon$, $f(y) - f_{xy}(y) = \epsilon$. By Theorem 1 we conclude that f is the uniform limit of functions in the closed ideal $\mathfrak{X}_0$ and hence that f is itself in $\mathfrak{X}_0$. We have now shown that $\mathfrak{Y}_0 \subset \mathfrak{X}_0$. Since $\mathfrak{X}_0 \subset \mathfrak{Y}_0$ by construction, we conclude that $\mathfrak{X}_0 = \mathfrak{Y}_0$ under the present hypothesis. Returning to the case where $\mathfrak{X}_0$ is an arbitrary non-void family, we consider a closed ideal $\mathfrak{X}_1$ containing $\mathfrak{X}_0$. Evidently $\mathfrak{X}_1$ has an associated function F_1 such that $F_1(x) \geq F_0(x)$ for every x. Hence $\mathfrak{X}_1 = \mathfrak{Y}_1 \subset \mathfrak{Y}_0$. Thus $\mathfrak{Y}_0$ is the smallest closed ideal containing $\mathfrak{X}_0$. With this the proof of the theorem is complete.

Next we shall consider the case where $\mathfrak{X}$ is treated as a linear lattice - the algebraic operations allowed including the linear operations as well as the two lattice operations. Here an ideal is to be defined as a non-void class closed under the allowed algebraic operations and enjoying the additional property that it contains with f every g such that $|g(x)| \leq |f(x)|$ for all x. Our principal result concerning closed ideals is essentially due to Kakutani [2]; it follows from Theorem 3 much as Theorem 6 follows from Theorem 2.

Theorem 7: *Let $\mathfrak{X}$ be the linear lattice of all the continuous real functions on a compact space X, let $\mathfrak{X}_0$ be an arbitrary non-void subfamily of $\mathfrak{X}$, let X_0 be the closed set of all those points x at which every function f in $\mathfrak{X}_0$ vanishes, and let $\mathfrak{X}_0$ be the family of those functions f in $\mathfrak{X}$ which vanish at every point of X_0. Then $\mathfrak{Y}_0$ is the smallest closed ideal containing $\mathfrak{X}_0$; and $\mathfrak{X}_0$ is a closed ideal if and only if $\mathfrak{X}_0 = \mathfrak{Y}_0$. A closed ideal $\mathfrak{X}_0$ is characterized by the associated closed set X_0; in particular, $\mathfrak{X}_0 = \mathfrak{Y}_0 = \mathfrak{X}$ if and only if X_0 is void.*

Proof: It is evident that $\mathfrak{Y}_0$ is a closed ideal containing $\mathfrak{X}_0$. For example, if f is in $\mathfrak{Y}_0$ and $|g(x)| \leq |f(x)|$ for every x, then g vanishes everywhere on X_0 and therefore belongs to $\mathfrak{Y}_0$. If $\mathfrak{X}_0$ is a closed ideal we can show that $\mathfrak{X}_0 = \mathfrak{Y}_0$. To do so we refer to Theorem 3 and consider what linear relations of the form indicated there can be satisfied by every function in $\mathfrak{X}_0$. Obviously the pairs of points x, y which have one or both members in X_0 are of no further interest, as the corresponding linear conditions are those of types (1) and (2), the effect of which has already been taken into account through the introduction of the closed set X_0. Turning to the case where x and y are distinct points not in X_0, we first remark that if we have $f(x) = f(y)$ for every f in $\mathfrak{X}$ then no *effective* restriction is implied by the linear relation corresponding to the pair of points in question. Assuming therefore that g is a function in $\mathfrak{X}$ with $g(x) \neq g(y)$, we may suppose without loss of generality that $g(x) = 1$, $g(y) = 0$ - for we may replace g if necessary by the function h defined through the equation $h(z) = \dfrac{g(z) - g(y)}{g(x) - g(y)}$ for all z in X. Since x is not in X_0 there is a function f in $\mathfrak{X}_0$ such that $f(x) \neq 0$. We may suppose without loss of generality that $f(x) > 1$ for we may replace f if necessary by the function $h = \alpha f$ with a suitable value of α. The function $h = |f| \cap |g|$ is now seen to be in the ideal $\mathfrak{X}_0$ and to satisfy the equations $h(x) = 1$, $h(y) = 0$. Accordingly no linear relation of type (3) or type (4) is satisfied by h. Hence the linear relations which characterize $\mathfrak{X}_0$ as a closed linear sublattice of $\mathfrak{X}$ reduce effectively to those implicit in the statement that every function in $\mathfrak{X}_0$ vanishes throughout X_0. It follows that $\mathfrak{X}_0 = \mathfrak{Y}_0$. Obviously if $\mathfrak{X}_0$ is an arbitrary non-void family and $\mathfrak{X}_1$ is a closed ideal containing $\mathfrak{X}_0$ then the associated closed set X_1 is part of X_0; and $\mathfrak{X}_1 = \mathfrak{Y}_1 \supset \mathfrak{Y}_0$. This completes the proof of the theorem.

Finally we take up the case where $\mathfrak{X}$ is to be regarded as a (linear) ring - the algebraic operations considered being addition and multiplication (and multiplication by real constants). Here an ideal is defined as a non-void subclass of $\mathfrak{X}$ which contains $f + g$ whenever it contains f and g, and which contains fg whenever it contains f. Since multiplication of f by the real number α is equivalent to multiplication of f by the constant function g everywhere equal to α, we see that an ideal automatically contains αf together with f. Our main result reads exactly like Theorem 7, differing from it only in the interpretation which has to be given to the term "ideal."

Theorem 8: *Let $\mathfrak{X}$ be the linear ring of all the continuous real functions on a compact space X, let $\mathfrak{X}_0$ be an arbitrary non-void subfamily of $\mathfrak{X}$, let X_0 be the closed set of all those points x at which every function f in $\mathfrak{X}_0$ vanishes, and let $\mathfrak{Y}_0$ be the family of all those functions f in $\mathfrak{X}$ which vanish at every point of X_0. Then $\mathfrak{Y}_0$ is the smallest closed ideal containing $\mathfrak{X}_0$; and $\mathfrak{X}_0$ is a closed ideal if and only if $\mathfrak{X}_0 = \mathfrak{Y}_0$. A closed ideal $\mathfrak{X}_0$ is characterized by the associated closed set X_0; in particular, $\mathfrak{X}_0 = \mathfrak{Y}_0 = \mathfrak{X}$ if and only if $\mathfrak{X}_0$ is void.*

Proof: It is evident that $\mathfrak{Y}_0$ is a closed ideal containing $\mathfrak{X}_0$. For example, if f is in $\mathfrak{Y}_0$ and g is arbitrary, then fg vanishes throughout X_0 and is therefore a function in $\mathfrak{Y}_0$. If $\mathfrak{X}_0$ is a closed ideal it is a closed linear subring. By using Theorem 5, we can show that $\mathfrak{X}_0 = \mathfrak{Y}_0$. The argument is much like that applied in the discussion of the analogous relation in Theorem 7. Since $\mathfrak{X}_0$ is characterized by the set X_0 and the linear relations of the form $g(x) = g(y)$ (where x and y are outside X_0) which are satisfied by every function in $\mathfrak{X}_0$, we have to eliminate the latter by appealing to the fact that $\mathfrak{X}_0$ is an ideal. If the condition $g(x) = g(y)$ is satisfied for *every* g in $\mathfrak{X}$, then there is no effective condition corresponding to the pair of points x, y. If there is some function g such that $g(x) \neq g(y)$, we may suppose that $g(x) = 1$, $g(y) = 0$. Since f can be found in $\mathfrak{X}_0$ so that $f(x) = 1$, the function $h = fg$ is in $\mathfrak{X}_0$ and $h(x) = 1$, $h(y) = 0$. Hence there is no condition of the form $g(x) = g(y)$ which is satisfied by all functions in $\mathfrak{X}_0$. This establishes the identity of $\mathfrak{X}_0$ and $\mathfrak{Y}_0$. The remainder of the discussion follows exactly the lines laid down in the proof of Theorem 7.

The connection between Theorems 7 and 8 is made plain by the following theorem.

Theorem 9: *If $\mathfrak{X}$ is the family of all continuous real functions on a compact space X, then $\mathfrak{X}$ is both a linear lattice and a (linear) ring. A non-void closed subfamily $\mathfrak{X}_0$ of $\mathfrak{X}$ is a linear-lattice ideal if and only if it is a ring-ideal.*

Proof: The result follows immediately from a comparison of Theorems 7 and 8. It would also be possible to give a proof by direct examination of the ideal-properties. Thus, if $\mathfrak{X}_0$ is a closed linear-lattice ideal, we show it to be a ring-ideal as follows. If f is in $\mathfrak{X}_0$ and g in $\mathfrak{X}$, then α can be found so that $|g(x)| \leq \alpha$ for every x - and it therefore follows that the product $h = fg$ satisfies the relations

$$|h(x)| = |f(x)g(x)| \leq |\alpha f(x)| = |(\alpha f)(x)|$$

and hence belongs to $\mathfrak{X}_0$ along with f and αf. On the other hand, if $\mathfrak{X}_0$ is a closed ring-ideal, it is a closed linear sublattice by virtue of Theorem 4. In particular, $\mathfrak{X}_0$ contains $|f|$ together with f. Thus, if f is in $\mathfrak{X}_0$ and g is in $\mathfrak{X}$, the function h_n defined by putting $h_n(x) = |f(x)|g(x)/\left(|f(x)| + \frac{1}{n}\right)$ for all x is in $\mathfrak{X}_0$. If g satisfies the inequality $|g(x)| \leq |f(x)|$ for every x, then

$$|g(x) - h_n(x)| \leq |g(x)|/(n|f(x)| + 1) \leq |f(x)|/(n|f(x)| + 1)$$

$$\leq \frac{1}{n}\left(1 - \frac{1}{n|f(x)| + 1}\right) \leq \frac{1}{n}.$$

Since the sequence h_n thus converges uniformly to g, we see that g is in $\mathfrak{X}_0$ also.

(*to be continued in the next issue*)

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CONGRUENCE METHODS AS APPLIED TO DIOPHANTINE ANALYSIS

by

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A fruitful method for the study of many mathematical questions is the application to them of the theory of arithmetical congruences, altho the questions themselves do not involve explicitly the latter concept. Among such applications is that of Hasse* to algebraic geometry and function theory. The writer** in the year 1940 set up an arithmetical theory of the Bernoulli numbers based on the idea that if $a - b \equiv 0 \pmod p$, with a, b, and p integers, and a and b independent of p, where p is any prime, then it follows that $a = b$, that is from certain congruences we may derive equations. On the other hand we may in a sense reverse this procedure by showing that if certain congruences are impossible then the corresponding equations are impossible, as was done in a recent paper of the writer's.† The point of view which was employed in the latter paper was a bit different from that used by previous investigators along these lines. The usual thing has been the examination of a single Diophantine equation such as

$$f(x,y,z) = x^4 + y^4 - 3z^4 = 0 \tag{1}$$

and using congruences with small moduli to show that it was impossible in integers x, y and z not all zero. For example it is quite easy to show†† that $f(x,y,z) \equiv 0 \pmod 3$ is an impossible congruence unless x, y and z are each divisible by 3, and consequently (1) is impossible for $xyz \neq 0$. Here we were very fortunate in that the first odd prime we tried as a modulus yielded our result. But we would never have proved that

$$x^3 + 3y^3 = 22 \tag{2}$$

is impossible by such a method, as the congruence

$$x^3 + 3y^3 \equiv 22 \pmod m \tag{3}$$

*H. Hasse, *Abh. Ges Wiss Gottingen.* Math-Phys. Kl., III. F. H. 18, 51-55 (1937).

***Proc. Amer. Math. Soc.*, 51, pp. 502-31 (1942).

†*Proc. Nat'l Acad. Sci.*, 32, pp. 101-6 (1946).

††By a method used to show that $x^4 + y^4 - 3z^4 \equiv 0 \pmod 5$ is impossible, $xyz \not\equiv 0 \pmod 5$, in last reference p. 102.

has solutions for every integral modulus *m*, yet, (2) in fact, is impossible. In direct contrast to this situation, Hasse** proved that the bilinear equation

$$f(x_i, x_j) = \sum_{i,j} a_{ij}\, x_i\, x_j = 0$$

is solvable in integers if and only if

$$f(x_i, x_j) \equiv 0 \pmod m$$

is properly solvable† for every integer *m*.

As in our previous paper, we do not start by considering a single Diophantine equation and applying congruence methods to it, *we proceed in a reverse fashion and begin by setting up classes of impossible congruences and inferring that the corresponding classes of equations have no solutions.* Also, we seek, and in some cases find, theorems which enable us to isolate impossible congruences with little of the element of trial being employed.

Consider first the congruence

$$(4) \qquad aX^m + bY^m + 1 \equiv 0 \pmod p,$$

$ab \not\equiv 0 \pmod p$; *p* prime; and consider the solutions X and $Y \not\equiv 0 \pmod p$. Further, let $p = 1 + cm$ and suppose that *g* is a primitive root of *p*. Then (4) may be written in the form

$$(5) \qquad 1 + g^{i+mx} \equiv g^{j+my} \pmod p,$$

where $g^i \equiv a$, $g^j \equiv -b$; $0 \leq i < m$, $0 \leq j < m$, and our problem is to find the possible sets *x*, *y* which satisfy this congruence for *i* and *j* fixed. Since $g^{p-1} \equiv g^{mc} \equiv 1 \pmod p$, it is sufficient to take $0 \leq x < c$; $0 \leq y < c$. Let the symbol (i,j) denote the number of different sets x,y which satisfy (5). Then we know†† that if $g^{\epsilon} \equiv -1 \pmod p$,

$$(6) \qquad \sum_{i=0}^{m-1} (i,j) = c \text{ for } j \not\equiv 0; \quad \sum_{i=0}^{m-1} (i,0) = c - 1;$$

*Skolem, *Avh. Norske Vid. Akad. Oslo* I, no. 4 (1942). The writer has been unable to consult this article, but has read a review of it in *Mathematical Reviews*.

***Crelle*, 152, pp. 129-148 (1923).

†Properly solvable means here that $(x_a, m) = 1$ for each a.

††Mitchell, *Annals Math.* 17, 165-177 (1916).

and

(7) $$\sum_{j=0}^{m-1} (i,j) = c, \text{ for } i \not\equiv \epsilon; \quad \sum_{j=0}^{m-1} (\epsilon,j) = c - 1;$$

it follows from these relations that if $c < m + 1$ then at least one of the (i,j) in (5) is zero. *Hence we are now in a position to set up, without trial, a number of impossible congruences, by taking various values of* $p = 1 + mc$, *choosing* $c < m + 1$ *and calculating directly the numbers* (i,j) in (5). For this purpose it is convenient to use Jacobi's* tables which give the values *h* and *k* in

(8) $$g^h + 1 \equiv g^k \pmod{p},$$

for each prime $p < 100$. For example $m = 4$, $c = 3$ and $p = 13$ in (5) give the result that ((0,3), (0,0), (1,2), (2,0), (2,2) and (3,1) are all zero when $g = 2$. The relation $(1,2) = 0$ shows that

(9) $$1 + 2u^4 \equiv 4v^4 \pmod{13}$$

has no solutions with $uv \not\equiv 0 \pmod{13}$. If $u \equiv 0 \pmod{13}$ then $v \not\equiv 0 \pmod{13}$ in (9) whence $4v^4 \equiv 1 \pmod{13}$ or $4^3 \equiv 1 \pmod{13}$ which does not hold. Hence $u \not\equiv 0 \pmod{13}$. Similarly if $v \equiv 0 \pmod{13}$ then $u \not\equiv 0 \pmod{13}$ whence $2^3 \equiv -1 \pmod{13}$ which is also impossible consequently $v \not\equiv 0 \pmod{13}$ in (9), hence that congruence has no solutions whatever, giving the result that

(10) $$1 + 2u^4 = 4v^4$$

is impossible, but this is an otherwise obvious result since the left-hand member is odd and the right even. However, if we multiply (10) by the integer 5 we obtain since $5 \equiv 31$, $20 \equiv 7 \pmod{13}$,

(11) $$7v^4 - 10u^4 \equiv 31 \pmod{13},$$

which is a congruence having no solutions since (9) has none. Consequently the equation

$$7v^4 - 10u^4 = 31$$

has no solutions, a result which does not appear trivial. In general if $s \not\equiv 0 \pmod{13}$ we have from (9) the result that

(12) $$(s + 13t_1) + (2s + 13t_2)\, u^4 = (4s + 13t_3)\, v^4$$

has no solutions for *s* any integer not divisible by 13 with t_1, t_2 and t_3 arbitrary integers. This result follows directly from the

**Crelle*, 30, pp. 181-82 (1846); *Werke* VI, 272-4.

fact that (9) is impossible since we obtain, if (12) does have solutions in u and v, using $s + 13t_1 \equiv s \pmod{13}$ etc.,

$$s + 2su^4 \equiv 4sv^4 \pmod{13}, \tag{13}$$

and multiplying thru by s_1, with $ss_1 \equiv 1 \pmod{13}$, we obtain (9). Using the other (i,j), s in (5) which were zero for $m = 4$, $c = 3$ we find that

$$s + 13t_1 + (s + 13t_2)\, u^4 = (8s + 13t_3)\, v^4$$

$$s + 13t_1 + (8s + 13t_2)\, u^4 = (2s + 13t_3)\, v^4$$

are impossible with s and the t's defined as before. (Induction indicates that congruences with no solutions may be found of the form (5), with m much less than c. For example it is known that, if $xy \not\equiv 0 \pmod{491}$,

$$x^7 + y^7 + 1 \equiv 0 \pmod{491}$$

is impossible.) Also if we take $h = 31$, $m = 5$ with $c = 6$, we find that for $g = 3$, (3,3), (4,4), (0,1), (0,2), (1,0) and (2,0) are all zero. The first two relations yield the result that

$$s + 31t_1 + (27s + 31t_2)\, u^5 = (27s + 31t_3)\, v^5$$

$$s + 31t_1 + (19s + 31t_2)\, u^5 = (19s + 31t_3)\, v^5$$

are all impossible for $s \not\equiv 0 \pmod{31}$ with s and the t's defined as before. In view of the above we have the Theorem I. *To find classes of trinomial equations with no solutions consider the congruence*

$$ax^m + 1 \equiv by^m \pmod{p} \tag{14}$$

for some prime p *of the form* $1 + cm$ *with* $c < m + 1$. *For at least one set of values of a and b (14) will have no solutions except possibly for x or y divisible by h. If (a,b) is a set such that (14) has no solutions then determine if an x exists with* $ax^m \equiv -1 \pmod{p}$ *and* y *such that* $by^m \equiv 1 \pmod{p}$. *If neither of these congruences is satisfied then each of the equations*

$$(sa + t_1p) + (s + t_2p)\, u^m = (sb + t_3p)\, v^m$$

is impossible in integers if s *is any integer prime to* p *and* t_1, t_2 *and* t_3 *are arbitrary integers.*

The method may be extended to congruences of the form

$$ax^m + by^m \equiv dz^m \pmod{p}, \tag{15}$$

$abd \not\equiv 0 \pmod{p}$. If we assume that $y \not\equiv 0 \pmod{p}$ then division by by^m gives

$$(16) \qquad a/b\, x_1^m + 1 \equiv d/b\, z_1^m \pmod{p}.$$

If however $y \equiv 0 \pmod{p}$ then it follows that

$$ax^m \equiv dz^m \pmod{p},$$

and if this congruence has no solutions aside from $x \equiv z \equiv 0$, then (15) has no solutions if also (16) has none, so in the latter event

$$ax^m + by^m = dz^m$$

is impossible in integers with $xyz \neq 0$, since, unless $xyz = 0$, we may assume that one of the integers $x, y, z \not\equiv 0 \pmod{p}$. For example, using (10), the equation

$$(17) \qquad 7v^4 - 31w^4 = 10u^4$$

is impossible, since, assuming $w^4 \not\equiv 0 \pmod{13}$ and dividing thru by $-31w^4$ we get

$$(18) \qquad 2u^4 + 1 \equiv 4v^4,$$

which has no solutions. If $w^4 \equiv 0 \pmod{13}$ then (17) gives

$$7v^4 \equiv 10u^4, \pmod{13},$$

which yields, assuming $uv \not\equiv \pmod{13}$, $7^3 \equiv 10^3 \pmod{13}$ which is impossible. We may state a theorem, analogous to Theorem I, to cover these homogeneous equations.

Multiplying impossible congruences such as (9) thru by an integer prime to the modulus is a special case of the result that such congruences have the *multiplicative* property that if $f_1(u, v) \equiv 0 \pmod{p}$ and $f_2(u^1, v^1) \equiv 0 \pmod{p}$ are impossible congruences in u, v, u^1, v^1, then $f_1 f_2 \equiv 0 \pmod{p}$ is also impossible. We shall now show that it is possible to obtain impossible equations by *addition* of a certain kind.

Let $f_i(u_1^{(i)}, u_2^{(i)}, \ldots, u_s^{(i)})$ be a polynomial in the *u*'s with integral coefficients, $i = 0, 1, \ldots, n-1$ and suppose that f_i homogeneous in the *u*'s for $i < n-1$ and is of degree $\geq n - i$, $i > 1$ and $f_i \equiv 0 \pmod{p}$, for $i < n-1$, has only the solutions $u_1^{(i)} \equiv u_2^{(i)} \equiv \cdots \equiv u_s^{(i)} \equiv 0 \pmod{p}$, and further that

$$(19) \qquad f_{n-1}(u_1^{(n-1)}, u_2^{(n-1)}, \ldots, u_s^{(n-1)}) \equiv 0 \pmod{p}$$

has no solutions, we may then show that the equation

$$(20)\qquad \sum_{i=0}^{n-1} p^i f_i(u_1^{(i)}, u_2^{(i)}, \ldots, u_s^{(i)}) = 0$$

has no integral solutions in the *u*'s. For if it has solutions, then reduction modulo *p* gives

$$f_o(u_1^{(0)}, u_2^{(0)}, \ldots, u_s^{(0)}) \equiv 0 \pmod p,$$

and by hypothesis it then follows that $u_1^{(0)} \equiv u_2^{(0)} \equiv \ldots \equiv u_s^{(0)} \equiv 0$ (mod p). Substituting these values in (20) gives, since f_o is homogeneous and of degree n,

$$(21)\qquad \sum_{i=1}^{n-1} p^i f_i(u_1^{(i)}, u_2^{(i)}, \ldots, u_s^{(i)}) \equiv 0 \pmod{p^n};$$

whence

$$f_1(u_1^{(1)} u_2^{(1)}, \ldots, u_s^{(1)}) \equiv 0 \pmod p.$$

Now f_1 is of degree $\geq n - 1$ and homogeneous, as well as solvable only for $u_1^{(1)} \equiv u_2^{(1)} \equiv \ldots \equiv u_s^{(1)} \equiv 0$ (mod p). Hence substitution of these values in

(21) gives

$$f_1(0,0,\ldots 0) \equiv 0 \pmod{p^{n-1}},$$

and

$$\sum_{i=2}^{n-1} p^i f_i(u_1^{(i)}, u_2^{(i)}, \ldots, u_s^{(i)}) \equiv 0 \pmod{p^n}.$$

Proceeding in this way we obtain finally the relation (19), but we stated initially that this congruence had no solutions hence (20) has no solutions in integers under the conditions stated concerning the *f*'s.

We now apply the same idea to homogeneous equations. Consider, where each *h* is a polynomial with integral coefficients,

$$\sum_{i=0}^{n-1} p^i h_i(x_1^{(i)}, x_2^{(i)}, \ldots, x_s^{(i)}) = 0, \tag{22}$$

and suppose that each h is of degree $t \geq n$ and that each $h_i \equiv 0 \pmod p$ has only the solutions $x_j^{(i)} \equiv 0 \pmod p$ with i any of the integers $0, 1, \ldots, n-1$ and for $j = 1, 2, \ldots, s$. Further not all the x's in (22) are zero. Then (22) has no integral solutions in the x's. For in the first place, if (22) has each x divisible by p^k, but not all divisible by p^{k+1}, then dividing said equation thru by p^{kt} we obtain

$$\sum_{i=0}^{n-1} p^i h_i(y_1^{(i)}, y_2^{(i)}, \ldots, y_s^{(i)}) = 0, \tag{23}$$

with not all the y's divisible by p. Modulo p, this yields

$$h_0(y_1^{(0)}, y_2^{(0)}, \ldots, y_s^{(0)}) \equiv 0 \pmod p,$$

and since by hypothesis this has no solutions in the y's except $y_1^{(0)} \equiv y_2^{(0)} \equiv \ldots y_s^{(0)} \equiv 0 \pmod p$ we find from (23)

$$\sum_{i=1}^{n-1} p^{i-1} h_i(y_1^{(i)}, y_2^{(i)}, \ldots, y_s^{(i)}) \equiv 0 \pmod{p^{n-1}},$$

giving $h_1 \equiv 0 \pmod p$ and proceeding as in our treatment of (21) we find finally that

$$h_{n-1}(y_1^{(n-1)}, \ldots, y_s^{(n-1)}) \equiv 0 \pmod p,$$

giving $y_1^{(n-1)} \equiv \ldots \equiv y_s^{(n-1)} \equiv 0 \pmod p$ which contradicts our hypothesis concerning (23), that not all the $y_j^{(i)}$; $i = 0, 1, \ldots, n-1$; $j = 1, 2, \ldots, s$ are divisible by p. Hence (23), and therefore (22) has no integral solutions unless the x's are all zero.

We may apply the method described in Theorem I to non-homogeneous equations.*
Consider

$$1 + ax^4 \equiv by^3 \pmod{13}.$$

*This problem is being investigated in general by Mrs. E. H. Pearson of Rice Institute.

Now 2 is a primitive root of 13, and if [i,j] represents the number of solutions α,β, in

$$1 + 2^{i+4\alpha} \equiv 2^{j+3\beta} \pmod{13}$$

we find that [0,0], [3,1], [2,2], [1,0] are all zero and using these we discover that

$$s + 13t_1 + (8s + 13t_2)\, u^4 = (2s + 13t_3)\, v^3$$

$$s + 13t_1 + (4s + 13t_2)\, u^4 = (4s + 13t_3)\, v^3$$

are all impossible in integers u and v if x is any integer $\not\equiv 0 \pmod{13}$, and t_1, t_2 and t_3 are arbitrary integers. For example the second relation gives for $s = 10$, $v^3 - u^4 = 10$.

Similarly, using the modulus 31 we find that none of the equations

$$s + 31t_1 + (as + 31t_2)\, x^6 = (bs + 31t_3)\, y^5$$

are solvable in integers x and y if a and b have any of the following sets of values:
1,9; 1,19; 9,3; 27,27; 19,9; 19,19; 26,3; 26,9; with s and the t's defined as before.

As is well known, Diophantine equations of higher degrees have been little studied. The methods of the present paper enable us to set up, extensive tables setting forth such Diophantine equations with no solutions. If this were done it might be valuable to a future investigator along these lines who, having possibly at hand much deeper methods than those we employ, would like to know in advance that certain classes of Diophantine equations were impossible of solution in integers, so that he would not have to try to apply his methods to them.

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the Mathematics Magazine, lists of new books, and book reviews.

The purpose and policies of the first division of this department (Comments on Papers) derive directly from the major objective of the Mathematics Magazine which is to encourage research and the production of superior expository articles by providing the means for prompt publication.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited. Comments which express conclusions at variance with those of the paper under review should be submitted in duplicate. One copy will be sent to the author of the original article for rebuttal.

Communications intended for this department should be addressed to

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The use of mathematics in government research. This comment by Kay L. Nielson of the Naval Ordinance Plant at Indianapolis is an interesting addition to the articles of Bollay and Sokolnikoff in the Nov. - Dec. issue.

The practice of employing mathematicians in industrial and governmental research, the need for changing the training of mathematicians to fill these positions and the possible opportunities in future years have all been rather widely discussed in recent years. Thus a symposium on these subjects was held at Pomona College on March 8, 1947.* An address on the "Industrial Mathematicians" was presented before the American Society for Engineering Education.** Universities such as Brown, New York University,

*See I. S. Sokolnikoff, "Opportunities for Mathematically Trained College Graduates," Mathematics Magazine, vol. XXI, No. 2., Nov.-Dec., 1947, pp. 102-105.

**K. L. Nielsen, "The Mathematical Engineer or the Industrial Mathematician," The Journal of Engineering Education, vol. 38, Nov. 1947, pp. 175-180.

Massachusetts Institute of Technology and others have established some graduate work in applied mathematics. The American Mathematical Society has created a Committee on Applied Mathematics.[†] Numerous other examples can be cited.

An interesting observation which has been made through attendance of these discussions, reading of reports of meetings and articles pertaining to this matter, yields the fact that the aircraft industry is used most often as an example[**†]. Whether it is the romanticism associated with aircraft or the fact that the industry does indeed make excellent use of mathematicians as such which prompts the use of this particular industry as an example, is difficult to ascertain. That this industry has an excellent outlet for mathematical research and presents a very promising future for mathematicians cannot be denied and from the viewpoint of mathematicians this industry should indeed be commended. Electrical and communications industries have also been called upon to furnish illustrative examples[††]. The recent rapid growth of the use of mathematical methods in quality control has called upon the manufacturers of precision instruments to emphasize desired arguments. However, in spite of the fact the statements are made with the phrase "industrial and governmental research," the *governmental* use of mathematicians has received little or no publicity in mathematical journals nor has it been called upon in discussions and addresses for illustrative purposes.

The assumption that it is public knowledge that governmental agencies employ mathematicians is utterly false. The employment by governmental agencies of mathematicians as mathematicians is in fact quite recent. They have, of course, always employed statisticians and this is the usual connotation given to any government employed mathematician. Quite frequently when mathematicians were employed in a capacity other than statistician they were classified as physicists. However, during and since the last war governmental agencies have employed a considerable number of mathematicians and have received specific classifications for them.

At the present time there exists a large number of research organizations operated by various agencies of the government who employ scientists of all kinds including a sizable number of mathematicians, Many of these organizations operate under the civil service plan and the scientists are classified under civil service professional ratings. All the rules and regulations of civil service pertaining to initial employment, promotions, retirement, clock hours, etc., including those which are detrimental to greater scientific endeavors, do apply to these organizations.

*Bulletin American Mathematical Society, vol. 53, No. 7., July, 1947, pp. 637-640.

**William Bollay, "The Use of Mathematicians in the Aircraft Industry," Mathematics Magazine, vol. XXI, No. 2., Nov.-Dec., 1947; pp. 105-109.

†K. L. Nielsen: "Industrial Experience for Mathematics Professors," American Mathematical Monthly, vol. LIV, No. 2., Feb., 1947; pp. 91-96.

††T. C. Fry, "Industrial Mathematics," American Mathematical Monthly, vol. 48, No. 61, Part II, Supplement, June-July, 1941.

Information on salary schedules, qualifications and experience which govern the professional ratings can, of course, be obtained from any civil service board. There are also a few scientific organizations which receive their major support from the government but which do not come under civil service.

The research conducted by these organizations depends upon the agencies with which it is affiliated. Many of the laboratories which come under the cognizance of the Army, Navy, or Air Force department conduct, of course, research connected with military problems and with the present scientific state of development of modern warfare this covers practically all fields of investigation. The late war also emphasized the need for more basic research and has thus led to the establishment of basic research projects. Consequently, the employment of mathematicians by governmental agencies is not always limited to the applied mathematician although he is heavily favored.

The primary point is that mathematical research is being conducted by governmental agencies. This research covers practically all fields and all levels. The government can and does offer a career for a mathematician. The usual approach to such a career is through the channels of civil service even though this may not be the most desirable approach. As mathematicians and as teachers of mathematicians perhaps we should be better informed about the opportunities offered by governmental agencies for mathematicians.

Analytic Geometry. By F. D. Murnaghan.
New York, Prentice-Hall, Inc., 1946. 8 + 402 pages. $3.25.

The first four chapters are used to introduce Cartesian reference systems, coordinates, rays, vectors, weighting coordinates of a point, for the one, two, and three-dimensional spaces. Such topics as linear dependence of vectors, scalar product, alternating product of three vectors, oriented triangles and tetrahedra are taken up in addition to the conventional subject matter. Matrices and determinants appear in the fifth chapter in connection with linear equations: we find column and row vectors, the sum and product of two matrices, rank, the adjoint and reciprocal of a matrix and orthogonal matrices. These five chapters cover the linear part of analytic geometry.

The remaining chapters take up in order applications of vector and matrix theory to the circle (including inversion), sphere, parabola, ellipse and hyperbola, and second degree surfaces (cylinders, cones, surfaces of revolution, ellipsoids, hyperboloids and paraboloids). The last two chapters give the reduction to canonic forms of the general second degree equations in two and three variables, respectively.

The author and the publishers are to be complimented on presenting a modern text. The increased use of vectors and matrices in the physical sciences appear to warrant the early introduction of these concepts in courses given by mathematicians. It is agreeable to read a book on anlytics which does not attemptto cover inadequately and at length topics which are better treated by the methods of calculus.

W. E. Pyrne

A collection of papers in memory of Sir William Rowan Hamilton. The Scripta Mathematica Studies No. 2. Published by Scripta Mathematica, Yeshiva College, New York, 1945. 82 pp.

The editors of Scripta Mathematica have published this collection of essays in commemoration of the one hundredth anniversary of Hamilton's discovery of the quaternions, which took place in 1843.* It contains a short estimate of Hamilton by D. E. Smith an appraisal of Hamilton's early work on optics and partial differential equations by J. L. Synge, who also gives a short biography, an article by C. C. Mac Duffee on Hamilton's work in algebra, an elementary presentation of quaternion theory by F. D. Murnaghan and a contribution by H. Bateman on Hamilton's work in dynamics and its influence on modern thought. There is a portrait of Hamilton with one of his sons, taken circa 1845, a picture of Trinity College a century ago and an announcement by the Irish minister of posts and telegraph concerning a special Hamilton postage stamp. Added is a short tribute to Hamilton by E. B. Wilson. A paper by V. Karapetoff on "The Constancy of the velocity of light," abridged from his book on the theory of relativity, has been added, though it seems to have no direct bearing on Hamilton's work.

Professor Synge, who is one of the editors of Hamilton's "Mathematical Papers" throws light on the importance of Hamilton's characteristic or principal function, which only received due recognition long after his death. Synge stresses the point that Hamilton's theory, in dynamics as well as in optics, transformed the problem of solving the ordinary differential equations of dynamics into that of solving two partial ones. This discovery has been somewhat obscured by the fact that Jacobi, applying Hamilton's theory, stressed the use of only one of these two partial differential equations - since known as the Hamilton-Jacobi equation. "But if your motive is to reduce the consideration of a dynamical system to that of one single principal function, then the second equation is fundamental." This remark of Professor Synge may help us to understand better Hamilton's original theory, which has a more general character than the Hamilton-Jacobi theory of our textbooks. We can gratefully grant Professor Synge this important

*It is essentially the same as Scripta Mathematica 10 (1944) pp. 7-80.

point without agreeing with him in his remark that "Jacobi belonged less to the heroic age than Hamilton." The difference between Jacobi & Hamilton seems rather to lie in the fact that Jacobi's work has been integrated in the whole heroic mathematical heritage of the first half of the nineteenth century, while Hamilton's work still struggles for the recognition it so well deserves.

Professor MacDuffee gives a clear and useful exposition of Hamilton's contributions to algebra, which lie especially in the clarification of the fundamental conceptions of algebra. Hamilton was one of the first to understand that algebra lacked an axiomatic, foundation and tried to construct one, though he hesitated to accept fully an abstract point of view. To the end of his career he felt it necessary to appeal to the physical universe in order to justify his abstract notions in the eyes of his fellow scientists. It might have been instructive if Professor MacDuffee had compared Hamilton's introduction of complex numbers in 1833 with the way in which Gauss introduced them at about the same time (1833) in his theory of biquadratic residues.

The other papers clarify some of Hamilton's ideas. Professor Murnaghan reconstructs the process which led Hamilton to his quaternions in the language of matrix theory. Professor Bateman explains how Hamilton's formulation of the principle of least action has influenced later thought from Jacobi and Lie to Hilbert and Birkhoff. Added are also two poems by Hamilton; from which we quote:

Let no desire of ease
No lack of courage, faith, or love, delay
Mine own steps, or that high thought-paven way
In which my soul her dear commission sees.

D. J. Struik

BOOKS RECEIVED

Theory of Functions (revised edition). By J. F. Pitt. New York, King's Crown Press, 1947. 200 pages.

Elements of Nomography. By R. D. Douglass and D. P. Adams. New York, McGraw-Hill Book Co., 1947. 209 pages.

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Euclidean Geometry, Its Nature and Its Use. By J. Herbert Blackhurst. Des Moines, Iowa, Garner Publishing Company, 1947. 208 pages.

HISTORY AND HUMANISM

Edited by
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Papers on the history of Mathematics *per se,* the part it has played in the development of our present civilization and its relation to other sciences and professions are desirable for this department. Send papers to G. Waldo Dunnington, Northwestern State College, Natchitoches, Louisiana.

MATHEMATICS AND THE SOCIAL SCIENCES*

by Maurice Frechet

In order to awaken interest in the largest possible circle of readers, without, however, completely avoiding mathematical precision, we shall defer the use of mathematical notations to the end of this article.

Before we begin our study of the application of mathematics to social sciences (such as economics, sociology, psychology, history, philosophy, etc.) we shall say a few words about mathematics itself.

Many distinguished persons, whose study of mathematics was limited to grammar school or high school, think that mathematics is a science which has attained a state of perfection and that it has solved all the problems which it has encountered. "Is there anything left to discover" is a question which mathematicians are frequently asked. Mathematicians encounter problems daily which they do not know how to solve. Moreover, solution of some of these may possibly require several centuries. The mathematicians also know that the various technologies present problems, the solution of some of which, will, perhaps, be the beginning of a new branch of mathematics. For others, mathematics is a dull, dry or stern science, foreign to any idealism and without connection with real life.

Applications. A better informed public knows that mathematics has rendered immense service to astronomy and physics. But, in general, they have no conception of the increasing number of sciences which are dependent on mathematics. For instance - stereochemistry and crystallography make use of geometry and the theory of groups. Physical chemistry rests upon the theory of differential equations. Genetics is founded on the theory of probability. Far from being limited to physics, the field of application of mathematics is ever increasing. As soon as it becomes

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necessary in a branch of knowledge to employ numbers, mathematical thinking becomes obligatory.

In the first stages of a developing science, verbal qualitative descriptions are given. Then on the basis of very rough hypotheses, a model of reality is built which reproduces, but very imperfectly, the reality. Slight or important successive changes in the basic hypotheses allow one thereafter to press nearer and nearer to reality. This work of adjustment can only be pursued by a more and more extensive use of higher and higher branches of mathematics, combined with more and more precise methods of measurement.

Evolution of Science. It is essential to understand that no empirical law, experimentally discovered, no general principle invented through theory, can give a final explanation of nature, that all these are to be later replaced by more complex but more precise laws, by less simple but more adequate principles. We know, for instance, that Boyle's law relating volume v, and pressure p, under constant temperature is merely a partial formulation, and that there are known to be expressions of the relation between v and p which are nearer to the facts than the too-simple equation $pv = \text{constant}$. We know, similarly, that in astronomy, Ptolemy's system was effective in explaining a number of celestial phenomena, but that when astronomical measurements became better, this system had to give way to that of Copernicus. The law of universal gravitation (according to which two material particles attract each other in inverse ratio of the squares of their distances) allows astronomers to predict celestial phenomena with great accuracy. However, we know now that in some cases, corrections for these computations are provided by the more complex theory of relativity.

Ought we to conclude from these facts that science is vain, since it denies itself and abandons its own discoveries? There is a too frequent tendency to believe and to assert that each new explanation of the world leads to the rejection of previous ones which should only be considered then as errors. This is wrong: no explanation is final, but each one contributes its part of the truth. In their succession they are like the successive sketches of the sculptor who prepares a bust in clay: a bowl is the first representation of the head; then - eyes, a nose, and ears improve the likeness. The result, with which the sculptor stops, will, however, only be a better representation. It will lack color, warmth, in one word - life, and not to be perfect shall, however, in no way cancel its value.

The Role of Hypotheses. These successive improvements of science are obtained through the use of more and more adequate hypotheses. Each appearance of a new system of hypotheses must be followed by verification of its consequences. Too often, people are tempted to think that if this verification is satisfactory, it is not only because the choice of these hypotheses was good but also that the hypotheses themselves were in exact conformance with the facts. Not at all. First, it is not known whether other hypotheses could not lead to results which are just as satisfying. Besides, it is important to keep in mind that hypotheses which are initially very rough and arbitrary may lead to good results. The reason is that generally the theory leads to formulas containing arbitrary constants, which have to be determined in each particular case. The theory has therefore only given the shape of the formula and in practice the choice of the values of these constants, called parameters, will be dictated by the observed data. For instance, there is a mortality law called Gompertz-Makeham's Law which, for adult ages gives excellent results and is much appreciated by the actuaries. Now this law has been obtained by starting from the following highly simplified hypotheses. Gompertz had assumed that the "force of life" decreases when age increases. Since the formula based on this assumption proved to be only an approximation to the real observed law, Makeham improved it by observing that amongst the causes of death it was necessary to take into account not only a variable physiological factor but also chances of accident which vary little with age. All this analysis is very schematic, and, besides, is still verbal. It may be mathematically translated in various ways. The two interpretations of the original assumption amount to asserting that the death rate at a given age is the sum of two terms, one reflecting "the force of life" which varies with age as a decreasing geometric proportion, and the other the chance of accident which should be constant.* Now these interpretations of the original assumptions are rather arbitrary ones which might well be replaced by similar but different interpretations which would lead to different results. You see here clearly that the good agreement which has been verified between formula and numerical table may be no guarantee of the exactness of the basic hypotheses. So much so that, often, the mathematical interpretation of the qualitative hypotheses is influenced, as is here clearly the case, by the

*In mathematical language, if l_x is the probability of a newborn living to the age x, we have

$$\frac{d}{dx}\log l_x = Gc^x + S,$$

where G and S are constants. Putting

$$\log g = \frac{G}{\log c} \text{ and } \log s = S,$$

and introducing an integration constant k, we thus get the so-called Gompertz-Makeham's Law:

$$l_x = ks^x g^{c^x}.$$

effectiveness in practice of the analytical or numerical computations which result from the interpretation; legitimate consideration from the practical point of view, but a consideration which has no relation whatsoever with the effort to obtain an adequate understanding of the concrete world.

Application to Social Sciences. We apologize for having recalled ideas which are not new when applied to the natural sciences but which are too often ignored when applied to other domains. It is, however, the same technique of building a science which we will meet again in the development of the so-called social sciences. The application of mathematics to the whole of one of these sciences would find limitations similar to those which are observed in sciences of nature, such as biology. Limitations which, moreover, are destined to be reduced little by little without expecting, or even wishing, to see them completely disappear.

In addition to the natural limitations of the application of mathematics to some branches of social science, there are other imaginary limitations which have been raised (and which we proceed to discuss).

A Priori Objections. For centuries the question was not how mathematics could help the social sciences. On the contrary, the introduction of mathematics into these sciences was at first held as something inconceivable, and later conceived as a scandal. Even now it is possible to find distinguished persons who still think undesirable or even dangerous the use of mathematics in social sciences.

Some of the Objections. Mathematical deductions start from data relatively simple and exactly perceived; in social sciences, on the contrary, we meet extremely complex and changing situations. When, in order to simplify the problem, we neglect some elements, there is a risk of introducing in the real situation, modifications of which the immediate consequences appear plausibly as unimportant, but whose remote results may be great without having been guessed as such.

The answer is very simple. The same difficulty exists in the sciences of nature and though it has certainly slowed down their progress, it has not stopped it. Take, for instance, astronomy, Theoretically, the flight of a fly should alter the foreseen movement of the celestial system. Conversely, the cosmic rays coming from the immense number of celestial bodies should act upon our perceptions. However, we neglect both influences. It is only by a constant and prolonged labor that man has been able to distinguish, in the midst of the infinite complexity of material facts, those which intervene in an appreciable way in such or such phenomena. Moreover, the number of facts which are taken into account always increases as the precision of measurement.

A more serious objection is to be found in man's freedom. For centuries no obstacle of that kind could be found in the sciences of nature. Laplace wrote one day:

Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective positions of the beings which compose it; if, moreover, this intelligence were vast enough to submit these data to analysis, it would embrace in the same formula both the movements of the largest bodies in the universe and those of the lightest atom: to it nothing would be uncertain, and the future as the past would be present to its eyes. The human mind offers a feeble outline of that intelligence in the perfection which it has given to astronomy. Its discoveries in mechanics and in geometry, joined to that of universal gravity, have enabled it to comprehend in the same analytical expressions the past and future states of the world system.*

On the contrary, in all human sciences any prediction can be altered by an unforeseen decision of a single individual. We certainly find here a new difficulty, generally absent in the domain of the sciences of nature. Let us note, however, that at the atomic level, the uncertainty principle of Heisenberg makes evident an unforeseen analogy between the sciences of nature and the sciences of man. Nevertheless, it remains true that the existence of man's freedom limits the applications of mathematics to the social sciences. The limitation is, however, far from being complete. We will say, later on, a word about this, when dealing with men in the aggregate. But even if we consider a single individual, man's freedom is itself limited by his intelligence. For, if a man has a free choice in his decisions, these latter are not independent of the material situations in which he finds himself. Seated near a fire, we feel free to put our feet nearer, but we do not think of actually thrusting them into it. At a barber shop, we know that the barber, armed with a razor, might, if he wanted to, cut our throat. Nevertheless, we do not hesitate to ask him to shave us. In other words, even in daily life, we know how to draw conclusions, which we consider as sufficiently assured, in circumstances where human freedom comes into play, where it could theoretically reduce our predictions to naught. In extraordinarily exceptional cases (through madness, vengeance, and so on), this actually does occur. Do not such exceptional cases occur in science or technique, as when, for instance, an earthquake may ruin the computations of an architect or of an engineer?

Thus, if we are ready to admit that human freedom detracts from the sureness of conclusions in the social sciences, we nevertheless maintain that it does not suffice to make impossible or useless the application of deductive and quantitative methods to these sciences. One of the essential qualities of the businessman and of the politician is to be able to guess, in given circumstances, the reactions of people with whom he is dealing. Why could not the scientist arrive at rules in these situations,

*Laplace, Theorie Analytique des Probabilites, Introduction; Oeuvres, t. 7 (Paris, 1886), p. 6.

through a systematic study which would take the place of the direct intuition of these people? But, some will say, even if it is possible to predict, to a certain extent, actions of a given individual, how can you dream of being able to do so in the much more complex case of a multitude of human beings? However, what happens is just the opposite. Accidental deviations from the predicted behavior of each individual of the group shall mutually the better compensate the more numerous the group is. The global result in individual decisions - very varied and even sometimes in absolutely opposite directions - can, thanks to that compensation, lead to consequences sometimes as certain as the best accepted laws of nature. Let us think, for instance, of the immense variety of motives which may push men to struggle for life, or on the contrary, to renounce life and disappear. From 100,000 adult individuals chosen at random at the beginning of a year, there might be in the course of this year, theoretically from 0 to 100,000 suicides. In fact, the observations in Prussia from 1869 to 1890 have shown that this number of suicides has been constantly between 10 and 20. This is a remarkable steadiness* and could legitimately be considered as quite unforeseen. But, we are now entering into the domain of applications to social sciences of two particular branches of mathematics: Calculus of Probability and Mathematical Statistics. These are two such important applications that we intend treating them, later on, in detail. But we thought it was necessary at least to mention them here.

Finally, others object that mathematics is using constantly the number notion, whereas social sciences treat of entities which escape any notion of measure. It is true that there also is found a source of limitations. However, let us state that the number notion possesses in the social sciences a wider place than is generally admitted at first sight. For, besides magnitudes which can be added, there are magnitudes of various intensities which can be compared, and ordered. These intensities may then be "graded," each through a number. Such is, for instance, the satisfaction produced in a given individual by the acquisition of one good or another. If, for instance, he is offered the choice between a new and a used bicycle, between a roasted chicken and one boiled carrot, he will generally prefer in both cases the first alternative. In other words, when numerical values are connected with the satisfactions produced by one or the other alternative, these values will be to a certain extent arbitrary ones; however, the first one shall be greater than the second one. We thus can imagine a numerical scale of satisfactions. For a given individual, to each good shall correspond a number arbitrarily chosen and having in itself, when considered alone, no absolute signifi-

*More remarkable still is the fact that, among these same suicides, the percentage of hangings has in four successive parts of this period passed through the extraordinarily closely neighboring values - 60.8; 60.3; 61.4; 60.3.

cance, but which, however, shall have to be chosen so that its numerical value should be greater than the one which corresponds to a good giving him a smaller satisfaction.

This notion of satisfaction, also called utility, desirability or ophelimity, plays a fundamental role in the modern theory of economics. It is specifically associated with a given individual and the utility of the same good varies from one man to another. It is through the mutual interplay of compensations and compromises between, on the one hand, utility scales of the members of a group, (which determine the "demand" of each type of merchandise) and on the other hand, the set of quantities of the various available goods, that has established the price scale, which, itself, is common to all members of the group.

Having thus answered diverse objections to the possibility of a proper use of mathematics in social sciences, and without having entirely rejected them, but reducing their range, we have as yet only discussed a priori objections. Is, however, mathematics in fact really used in these sciences, and, a much more important question, is it used therein in an acceptable and useful way?

The Fact. It is curious to note that many of those who insisted on the inconceivability of using mathematics in the social sciences have actually made constant use of it in their own work. Economists never have been able to treat of their science without speaking of prices, that is, of numbers, to consider purchase or renting of land without taking into account its area; study the fortune of an individual without adding the various items of his fortune. All this means that they have utilized among other things arithmetical notions of number, its decimal representation, the adding operation and also the geometrical notion of area. All this from the very birth of economics. In more modern times they have besides, made more and more use of graphical representations to study, for instance, the variation of prices, consumption, and production, with time.

These economists might have thought that number, decimals, addition, area, graphical representation are so common, such widespread notions, that it might appear as an insult to mathematicians, should these notions be ascribed to mathematics.

However - and though this is often lost sight of, - they were mathematicians and of high mental capacity - though they have remained as unknown to us as the genial inventor of fire - these who, by a delicate and long work of generalization, of induction as well as deduction, have been able to invent empirical processes, and then from them evolve the rules which now allow even a child to represent numbers, to add them, to compute the area of a rectangle. It was necessary to wait until the year 1,000 in order to make the use of the Arabic figures general throughout Europe. We had to wait until the seventeenth century for the systematic use of the four fundamental arithmetical operations, and finally we had to wait until the eighteenth century for the square root process.

And it is to one mathematician, also known as one of the greatest philosophers, to Descartes, that we owe the invention of Analytic Geometry, and, as a consequence, the use of graphs to to represent numerical relations between two variables.

Reduction of the Question. Our last question becomes now only a matter of degree. The social sciences not only *can* use use mathematics, they *have* done it and this use is for them useful and indispensable.

There remains only to see what is the extent of the mathematical technique which, outside of the most elementary notions, can be usefully employed. We will examine this problem in detail later. We will, however, give an instance which shall show immediately in which direction the answer can be expected.

Mathematics and History. What relation could be anticipated between history and so-called differential equations? It is clear that a historian's conclusions about the action of a people or of a great man may be altered when considering a fact as a consequence of another fact, it happens that the sequence of these two facts is the reverse of what he thought it to be. To date events, exactly, is then, a first duty of any historian. Now, it appears that difficult problems of dates in ancient times have been successfully solved by noting that they had happened simultaneously with some astronomical phenomena (eclipses, and so on). Astronomy has determined the dates of these phenomena. But how? In most cases, there were, in those ancient times, no astronomers on the spot to observe them; their dates have been gotten by computations. Accordingly these computations are based, on the one hand, on very precise observations of analogous phenomena, which have appeared at modern times, and on the other hand, on the entirely mathematical theory of universal gravitation, a theory which relates the known dates of modern observations to the unknown dates which had to be determined. Now, this theory goes far beyond the elements of mathematics and, in particular, makes use of the previously mentioned differential equations.

Mathematics and Philosophy. Another instance of the application of relatively advanced mathematics in the social sciences is to be found in Philosophy in the study of Logic. They are amongst the most modern mathematical thinkers, those who have been able to show that logical reasoning can be expressed by the use of a very limited number of symbols such as $\rightarrow$, ϵ, $\subset$,.....(which mean respectively: implies, is an element of, is a part of,.....). They have also discovered a number of formal laws of Logic, thus establishing what is known as the "Algebra of Logic." At this point let us note that the social sciences derive a very considerable indirect benefit from the use of mathematics.

Indirect Influence. We have seen that mathematicians and specialists in other sciences who were provided with a mathematical culture have utilized relatively advanced mathematics in the study of the sciences of man. Now, it is interesting to note that side by side with the precise results which they could thus obtain, they have introduced thereby a method the spirit of which exerted an *indirect* influence over the specialists in the social sciences who were not proficient in mathematics. Thus, in economics, mathematical economists - who have been brought up in the study of functions and of equations - have naturally devoted themselves to the study of the relations of dependence between economic quantities. As a result, though formerly the classical economists had been chiefly busy in the research of causes, the same classical economists have, later on, been more frequently than before, interested in the simultaneous and interdependent variations of two - or more than two - economic quantities, even when they ignored the details of the works of the mathematical economists.

Finally, since the use of mathematics demands a rigorous and precise language, it has led those who were using it to introduce in definitions and statements concerning social sciences, greater rigor and precision, the effect of which was felt by those who use non-mathematical language. To illustrate this point I will refer you to the answers to an international inquiry, which we were asked to summarize by the office of the International Institute of Statistics in order to present a report at the Washington session in September, 1947, of this Institute (a report which has, moreover, been published, with the full text of the answers which it was summarizing, in Revue de l'Institut International de Statistique, 1947). I will quote here only one paragraph of that paper:

"At the end, say C. and Ed. Guillaume, mathematics is nothing other than an abridged language which employs symbols in order to convey ideas, which might be expressed in common language, were it not for their extreme complexity. The precision of the definitions which fix the meaning of the mathematical terminology makes it possible to maintain a rigor, which is excluded in the current language by the inevitable uncertainty in the comprehension of of terms which are used in multiple and ceaselessly varied meanings."

Limitations. In any application mathematics can do much, but it cannot do everything. It cannot do it, first of all from internal considerations. At each epoch, mathematics was developed in terms of problems which it could solve. The number of solvable problems has constantly increased, but at the same time the number of those which mathematics could formulate and in front of which it was powerless, has also constantly increased, primarily because of the increasing number of requests of the various techniques. Another source of difficulty has been that when a mathematical theory appears able to explain qualitatively a human phenomenon, it has to be verified quantitatively and therefore numerical data are indispensable. Now in many human sciences these data are completely lacking, and in others, like economics, it is only in recent times that the gathering of numerical, carefully established observations was started. This beginning of quantification is precisely what has permitted in an even more recent time, the birth of a new science "econometry."

Even when proper mathematical tools and numerical data are available, this is not sufficient to assure a satisfying result. Everything depends on the choice of the hypotheses which have been assumed in order to transform a human problem into a mathematical one. Classical economists are often tempted to ascribe to the use of mathematics, the failure of a mathematical theory of some economic phenomenon. Save for the rather rare case when the theory has been developed by a mathematical apprentice, this failure is not chargeable to the mathematics but to the bad choice of the assumed hypotheses. Of course, these economists would be right if it happened that no economic hypothesis is simple enough to be translated mathematically and at the same time close enough to daily life for the consequences to be valid with the same degree of closeness - but this is not the case and the success of econometry makes it the "flying wing" of political economy.

However, the intervention of human freedom introduces a factor which neatly differentiates the human sciences from the sciences of nature. Paradoxically enough, it can be said that human sciences constantly risk becoming falser precisely at moments when they are tending to become more true. If, in fact, a phenomenon having been studied, it is demonstrated that in given circumstances man naturally reacts in a certain way, then as soon as the law is well established, each individual, who is familiar with the law, will react, if the anticipated consequence is unfavorable to him, so as to avoid this consequence. Assume for instance that we learn that the occurrence of sun spots has until now, coincided with an increase of juvenile criminality. Is it not reasonable to presume that the state and the private social institutions would then take measures in order to counteract the foreseen effect and thus render the law false? Assume similarly, that a periodicity of past economic crises has been noted. If, for instance, we should thus have to foresee in this way a crisis in 1952, surely the state and the bankers would take measures before 1952 to avoid a crisis, for instance, in reducing the credits that they should normally have granted.

A Last Limitation. As a Science develops, some of the initial hypotheses are found to entail more and more remote consequences, which appear less and less acceptable. For instance, economic theory is actually based on the hedonistic principle, that is: that anybody tries to realize to the maximum what he thinks is his own interest. But psychology and sociology invite us to reconsider that principle. Is man always acting selfishly? Does he not, on occasion, sacrifice his life...and with it the enjoyment of his material goods - to a principle, an idea, or a person?

This must not, however, make us abandon the principle of maximum satisfaction. We may retain the hedonistic principle by including spiritual value along with material goods. Then the total satisfaction at a given time for a given individual shall still be a quantitative function. However, the quantities measured should include not only material, but also spiritual goods, such as: the pleasure of helping Smith, the pleasure of taking vengeance upon Brown, and so on. This assumes that we could assign figures grading these various satisfactions. This assumption might appear as very bold. Can we not, however, grade spiritual goods? After all, when we agree to pay up to a certain amount of money and not more, to hear a Mozart symphony or to listen to a Shakespeare play, and we thus set a maximum to our appreciation of an intellectual pleasure, have we not solved a problem of the same kind? Will not psychology coupled with the notion of probability, show us how to get out of this difficulty? It is, in any case, actually a difficulty which it was necessary to mention. Actually also, in the very rough state of our economic and social predictions, we can be reassured: that we are still in a situation where the present form of the hedonistic principle will help us during a long period.

Introduction of Literal Formulae. Complexity of the elements intervening in human sciences has been invoked against the use of mathematics in these sciences. In a way, it is, on the contrary, a circumstance which suggests the use of mathematics. It is precisely an advantage of mathematics to be able to lend itself to the simultaneous consideration of a great number and even of an infinite number of distinct elements. One succeeds in doing this by the use of appropriate notations. For instance, we can represent by p_1, p_2, ..., p_n, the prices of any number n of distinct commodities. Coming back to the notion of satisfaction above considered, let us denote by S the satisfaction produced by the set of commodities possessed by, say Smith, for instance a quantity measure q_2 of a commodity at price p_1, a quantity measure q_2 of another commodity at price p_2, ... Then S shall depend on q_1, q_2, ... and we will assume that it is a "function" of q_1, q_2, ... that is, that when the values of q_1, q_2, ... are given, then S is determined. And we will express that fact by the notation

$$S = f(q_1, q_2 ...)$$

which is read as: S equals f of q_1, q_2, ... Looking at this formula, any student of mathematics will understand that there is a relation between S and q_1, q_2 ... and even that the knowledge of q_1, q_2, ... is assumed to determine S. And in the sequence of his computations he does not need (as he would in common language) to replace each time the short notation q_2, for instance, by the sentence "quantity of the second commodity possessed by Smith."

Equations of Economic Equilibrium. Assume that we have been able to write several relations between a number of economic magnitudes in the equilibrium state. It is a well known scientific fact that if the number (say e) of the relations (also called equations) is greater than the number (say i) of the unknowns (that is, the values of these magnitudes), then one at least of these relations is necessarily a consequence of the others and might be dispensed with. And if e is smaller than i, then it can be shown that the number of equations does not suffice to determine the values of the unknowns. It is precisely in the application of this fundamental mathematical result (of which the statement should be somewhat restricted, but this would lead us too far) that rests one of the most important contributions of mathematical economists.

Example of Mathematical Reasoning in Econometry. Except in our instances of applications to History and Philosophy, we have not gone further in economics than the use of elementary mathematics, and we have not shown how to express the hypotheses from which are derived the above mentioned equations. It is not the proper place to enter into the details of the general theory which has been suggested above. We can, however, give an idea of the direction to follow by considering a simple example.

Suppose that our Smith may avail himself of a given capital, say C, in order to acquire the set of his future commodities. What shall the amounts q_1, q_2, ... of each of these commodities which he will finally decide to buy? It will be the system of values of q_1, q_2, ... which will give him the maximum satisfaction. These quantities have to be bought at a total price obviously equal to $p_1q_1 + p_2q_2 + \ldots$ and also to the total capital C. We are thus led to transform the economic problem into a mathematical problem, which we know how to solve, namely, to determine the values of q_1, q_2, ... which make maximum the function $S = f(q_1, q_2 \ldots)$ among those values of q_1, q_2, ... for which the following holds:

$$p_1q_1 + p_2q_2 + \ldots = C$$

There is a regular method to solve that problem, the method of "Lagrange's multipliers," a method known for a long time, but which still does not belong to the elementary part of mathematics.

It has been proved that we have to multiply the first member of the condition, written in the form

$$p_1q_1 + p_2q_2 + \ldots - C = 0$$

by a number λ, to add the product to $f(q_1, q_2, \ldots)$ and to assume that the required values of q_1, q_2, ... are those which render maximum the sum

$$f(q_1, q_2, \ldots) + \lambda(p_1q_1 + p_2q_2 + \ldots - C)$$

when q_1, q_2, vary quite independently. In order to do that, we must have recourse to Differential Calculus. According to that calculus, we must differentiate the same sum relatively to q_1, q_2, ... and set the derivatives of this sum equal to zero. This is expressed by equations (where the notations, such as $\frac{\delta f}{\delta q_1}$ shall be explained below):

$$1)\qquad \frac{\frac{\delta f}{\delta q_1}}{p_1} = \frac{\frac{\delta f}{\delta q_2}}{p_2} \cdots = \frac{\frac{\delta f}{\delta q_n}}{p_n}$$

$$p_1q_1 + p_2q_2 + \ldots = C.$$

We verify that we have here the required number, n, of equations (between the same number n, of the unknowns q_1, q_2 ... q_n) which is just sufficient to determine q_1, q_2, ... q_n.

It is interesting to note that these equations invalve expressions such as $\frac{\delta f}{\delta q_1}$, called "partial derivatives" of f with respect to q_1. This quantity may be intuitively defined as the rate of variation of f, that is of the satisfaction S, when q_1 alone varies. This rate is the ratio $\frac{\sigma}{r}$ where a given small increase r of q_1 corresponds to an increase σ of the satisfaction. When the unit of measurement of the first commodity is small enough, we can take r = 1 and $\frac{\sigma}{r}$ shall reduce to σ: Thus $\frac{\delta f}{\delta q_1}$ is the increase σ of f, which occurs when increasing the quantity q_1 by one unit.

Finally $\frac{\delta f}{\delta q_1}$ is equal to the increase in the satisfaction of Smith when he receives the last unit of the first commodity he possesses. That is why $\frac{\delta f}{\delta q_1}$ is also called the marginal satisfaction (or the marginal utility) of the first commodity possessed by Smith.

This definition being given, we may express the equations (1) by saying that: when Smith avails himself of a given capital to buy some goods at fixed prices p_1, p_2, ..., his choice will be such that the quantities of goods which he will buy are those which make the marginal utilities of these goods proportional to their prices.

It is interesting to note, by the way, that the interest of the notion of marginal utility was discovered independently and at about the same time by the classical economists of the Vienna school and by the first mathematical economists.

University of Paris

TEACHING OF MATHEMATICS

Edited by

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This department invites articles on methods of teaching, adaptation of subject matter and related topics. Papers on such topics or on any other subject in which you as a teacher are interested, or questions which you would like to have our readers discuss, should be sent to Joseph Seidin, Alfred University, Alfred, New York.

FUNDAMENTALS OF BEGINNING ALGEBRA

by

E. Justin Hills

Mathematics is a language just as Spanish, for example, is a language. If we should live in Spain for any length of time we would want to learn the language of the people so that we could get along better with them and enjoy their customs and methods of living. If we want to know mathematics so that we can get a better understanding of the subject and its marvelous accomplishments.

To learn Spanish we must acquire a vocabulary and learn to translate from English to Spanish and vice versa. To learn mathematics we must become familiar with the notation used and learn to put English expressions into mathematical form and vice versa.

Arithmetic, the art of computation, is the foundation on which all mathematics is based. In arithmetic we make use of numbers made up of numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

Algebra is a generalization of arithmetic. In algebra we make use of letters. Consider the sum of two numbers such as:

$$3 + 5, \text{ or } \quad 4 + 7.$$

Using letters, these sums can be denoted by the one expression

$$a + b$$

where a and b can be either 3 and 5 or 4 and 7.
Such a combination of letters tells briefly what is to be done to any pair of numbers that are under consideration.

Obviously *a* and *b* could be any numbers whatever, perhaps both the same number. Thus the algebraic statement, a + b, is a precise, condensed way of saying "Any number is to be added to any other or the same number."

Algebraic statement of multiplication

Since mathematics has a language of its own, we learn to translate English statements into mathematical form.

1. *English statements* — *Arithmetic statements*

English statements	*Arithmetic statements*
Three times five	3 x 5
Three times seven	3 x 7

Algebraic statement

3 x a, or merely 3a*

Then if a = 5, 3a = 15; or if a = 7, 3a = 21; etc.

2. *English statements* — *Arithmetic statements*

English statements	*Arithmetic statements*
Four times three	4 x 3
Six times seven	6 x 7

Algebraic statement

a x b, or merely ab*

Then if a = 4 and b = 3, ab = 4 x 3 = 12; or if a = 6 and b = 7, ab = 6 x 7 = 42; etc.

Further simplification by use of algebraic language will now be considered.

Multiplication and addition

English statements	*Arithmetic statements*
Eight more than five times three	5 x 3 + 8
Seven more than five times two	5 x 2 + 7

Algebraic statement

5a + b

Then if a = 3 and b = 8, 5a + b = 5 x 3 + 8 = 23; or if a = 2 and b = 7, 5a + b = 5 x 2 + 7 = 17; etc.

Multiplication and subtraction

English statements	*Arithmetic statements*
Four less than three times two	3 x 2 - 4
Seven less than three times five	3 x 5 - 7

Algebraic statement

3a - b

Then if a = 2 and b = 4, 3a - b = 3 x 2 - 4 = 2; or if a = 5 and b = 7, 3a - b = 3 x 5 - 7 = 8; etc.

Addition of products

English statements	*Arithmetic statements*
Two times three plus three times four	2 x 3 + 3 x 4
Two times four plus three times six	2 x 4 + 3 x 6

Algebraic statement

2a + 3b

Then if a = 3 and b = 4, 2a + 3b = 2 x 3 + 3 x 4 = 18; or if a = 4 and b = 6, 2a + 3b = 2 x 4 + 3 x 6 = 26; etc.

Subtraction of products

English statements	*Arithmetic statements*
Three times four minus two times five	3 x 4 - 2 x 5
Three times two minus two times one	3 x 2 - 2 x 1

Algebraic statement

3a - 2b

*Omission of signs between two letters or a number and a letter means multiplication, but cannot be used between two numbers.

Then if a = 4 and b = 5, 3a - 2b = 3 x 4 - 2 x 5 - 2; or if a = 2 and b = 1, 3a - 2b = 3 x 2 - 2 x 1 = 4; etc.

Products of products

English statements	*Arithmetic statements*
Four times two multiplied by two times three	(4 x 2) x (2 x 3) = 4 x 2 x 2 x 3
Four times three multiplied by two times one	(4 x 3) x (2 x 1) = 4 x 3 x 2 x 1

Algebraic statement

(4 x a) x (2 x b)
= 4a x 2b

Since a and b represent numbers, we can rearrange the numbers and letters as we please, That is

4a x 2b = 4 x 2 x a x b

Having done so, put down the product of the numbers. Therefore

4a x 2b = 8 x a x b, or 8ab

Then if a = 2 and b = 3, 8ab = 8 x 2 x 3 = 48; or if a = 3 and b = 1, 8ab = 8 x 3 x 1 = 24; etc.

Quotients of products

English statements	*Arithmetic statements*
Two times twelve divided by three times four	$\frac{2 \times 12}{3 \times 4}$
Two times nine divided by three times two	$\frac{2 \times 9}{3 \times 2}$

Algebraic statement

$$\frac{2a}{3b}$$

Then if a = 12 and b = 4, $\frac{2a}{3b} = \frac{2 \times 12}{3 \times 4} = 2$; or if a = 9 and b = 2, $\frac{2a}{3b} = \frac{2 \times 9}{3 \times 2} = 3$; etc.

Multiplication of a sum

English statement

Three times the quantity four plus seven; that is, three times the result of adding four and seven. Note in the first arithmetic statement below that this is the same thing as the sum of three times four and three times seven. And it will be found that this statement is true regardless of what numbers one would use in place of three, four, and seven. Note the second and third arithmetic statements.

Arithmetic statements

1) 3(4 + 7) = 3 x 11 = 33, or
= 3 x 4 + 3 x 7 = 12 + 21 = 33

2) 5(3 + 2) = 5 x 5 = 25, or
= 5 x 3 + 5 x 2 = 15 + 10 = 25

3) 4(3 + 2) = 4 x 5 = 20, or
= 4 x 3 + 4 x 2 = 12 + 8 = 20

The following algebraic statement represents these examples and any similar ones.

Algebraic statement

$$a(b + c) = ab + ac$$

Then if $a = 3$, $b = 4$, and $c = 7$,

$a(b + c) = 3(4 + 7) = 3 \times 11 = 33$, or

$a(b + c) = ab + ac = 3 \times 4 + 3 \times 7 = 12 + 21 = 33$

Then if $a = 5$, $b = 3$, and $c = 2$,

$a(b + c) = 5(3 + 2) = 5 \times 5 = 25$, or

$a(b + c) = ab + ac = 5 \times 3 + 5 \times 2 = 15 + 10 = 25$

Then if $a = 4$, $b = 3$, and $c = 2$,

$a(b + c) = 4(3 + 2) = 4 \times 5 = 20$, or

$a(b + c) = ab + ac = 4 \times 3 + 4 \times 2 = 12 + 8 = 20$

Sentences translated into algebraic language

Problems in arithmetic can be expressed in sentences. They can then be translated into algebraic language. Some illustrations are:

1. Some number plus four equals seven.
2. Some number minus two equals one.
3. Three times some number equals six.
4. One half of some number is two.

These sentences can be translated into algebraic language by letting some letter (usually a letter at the end of the alphabet) represent the number that is unknown. Below the letters x, y, and w have been chosen. Please note that such a choice is entirely arbitrary. Therefore the translations can be:

1. $x + 4 = 7$
2. $y - 2 = 1$
3. $3w = 6$.
4. $\frac{x}{2} = 2$

These statements in algebraic language are known as equations. An equation is the equality of two quantities. For example, in $x + 4 = 7$, the quantity $x + 4$ equals the quantity 7.

By trial, the values of the unknown number of an equation can be found. Consider the equations just presented.

1. $x + 4 = 7$

Value of x	Value of left side	Value of right side	Does left side equal right side
1	1 + 4 = 5	7	No
2	2 + 4 = 6	7	No
3	3 + 4 = 7	7	Yes
4	4 + 4 = 8	7	No

Therefore 3 is the only value of the letter that makes both sides (quantities) have the same number value. That is, for $x + 4 = 7$, if $x = 3$, then

$$3 + 4 = 7$$
$$7 = 7$$

2. $y - 2 = 1$

Value of y	Value of left side	Value of right side	Does left side equal right side
4	$4 - 2 = 2$	1	No
3	$3 - 2 = 1$	1	Yes
2	$2 - 2 = 0$	1	No

Therefore 3 is the only value of the letter that makes both sides have the same number value. That is, for $y - 2 = 1$, if $y = 3$, then

$$3 - 2 = 1$$
$$1 = 1$$

3. $3w = 6$

Value of w	Value of left side	Value of right side	Does left side equal right side
1	$3 \times 1 = 3$	6	No
2	$3 \times 2 = 6$	6	Yes
3	$3 \times 3 = 9$	6	No

Therefore 2 is the only value of the letter that makes both sides have the same number value. That is, for $3w = 6$, if $w = 2$, then

$$3 \times 2 = 6$$
$$6 = 6$$

4. $\frac{x}{2} = 2$

Value of x	Value of left side	Value of right side	Does left side equal right side
1	$1 \div 2 = \frac{1}{2}$	2	No
2	$2 \div 2 = 1$	2	No
3	$3 \div 2 = 1\frac{1}{2}$	2	No
4	$4 \div 2 = 2$	2	Yes

Therefore 4 is the only value of the letter that makes both sides have the same number value. That is, for $\frac{x}{2} = 2$, if $x = 4$, then

$$\frac{4}{2} = 2$$
$$2 = 2$$

These four illustrations show that there is only one arithmetic number for the letter in each of these equations. That is, the letter represents only one arithmetic number.

The equation is the most useful and important concept in algebra. It clarifies the solution of many problems that are practically impossible without the condensed language of algebra.

Illustrations:

1) An estate of $3000 is to be divided between a son and a daughter. The son's share is to be twice the daughter's share.

How much does each receive?

Arithmetic: Assume there are three shares, two for the son and one for the daughter. The three shares equal \$3000, so one share equals \$1000. Therefore the son's share is \$2000 and the daughter's share is \$1000.

Algebra: Let $x =$ the daughter's share.

$2x =$ the son's share.

$$x + 2x = \$3000$$

$$3x = \$3000$$

$x = \$1000$, the daughter's share (See p. 56.)

$2x = \$2000$, the son's share.

2) Four men earn jointly \$80 a day. Two of the men earn the same amount. The third man earns twice as much as the first or second; the fourth man earns as much as all the others. How much does each man earn per day?

Arithmetic: Let the earnings of the first two men be considered as one part (of the \$80) each. The third man then earns four parts of the same size. Therefore there are eight equal parts; so that each part of \$80 is \$10. Thus the first and second men earn \$10 each; the third man earns \$20; and the fourth man earns \$40.

Algebra: Let $x =$ part earned by the first or second man. Then the third man earns $2x$ and the fourth man earns $4x$. Therefore:

$$x + x + 2x + 4x = 80$$

$$8x = 80$$

$$x = 10$$

3) 25% of some number is 40, what is the number?

Arithmetic: $\dfrac{40}{25\%} = \dfrac{40}{0.25} = 160$

(This is a seventh grade problem but difficult without algebra.)

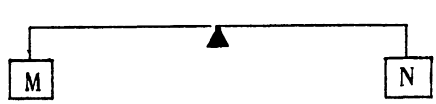
Algebra: Let $n =$ the number. Then $0.25n = 40$. And $n = \dfrac{40}{0.25}$

(See p. 56.)

Solution of equations

Many of the rules of algebra and the principles underlying them are based on the need to find the value of the unknown of equations, which is called solving for the value of the unknown. The trial method of solving equations (See p. 53) is too tedious. Faster and more interesting methods will now be explained.

Observe that an equation is like a balanced pair of scales. The value on the left side of the equal sign balances the value on the right side. Thus, if the pair of scales has equal arms,

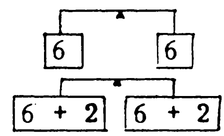


the value M on the left side equals the value N on the right side. Furthermore, the balance is maintained if the same amount is added to, or subtracted from, both sides, or if both sides are multiplied by, or divided by, the same amount. Let us state and illustrate with numbers each of these facts, usually called axioms.

1. An equality remains if the same amount is added to both sides.

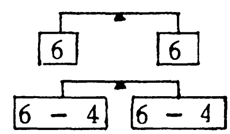
Illustration

$$6 = 6$$
$$6 + 2 = 6 + 2$$
$$8 = 8$$



2. An equality remains if the same amount is subtracted from both sides.

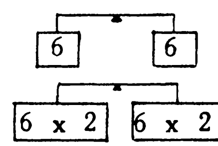
$$6 = 6$$
$$6 - 4 = 6 - 4$$
$$2 = 2$$



3. An equality remains if both sides are multiplied by the same amount.

Illustration

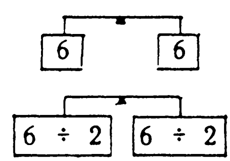
$$6 = 6$$
$$6 \times 2 = 6 \times 2$$
$$12 = 12$$



4. An equality remains if both sides are divided by the same amount.

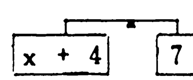
Illustration

$$6 = 6$$
$$6 \div 2 = 6 \div 2$$
$$3 = 3$$



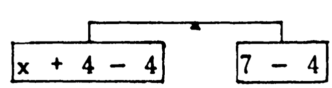
In the following equations, the unknown can be determined by using one or more of the above mentioned rules.

Given: $x + 4 = 7$



Subtract 4 from both sides in order to get x alone.

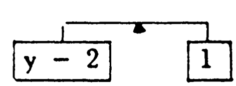
$$x + 4 - 4 = 7 - 4$$



Therefore:

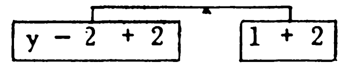
$$x = 3$$

Given: $y - 2 = 1$



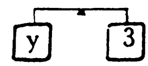
Add 2 to both sides in order to get y alone.

$$y - 2 + 2 = 1 + 2$$

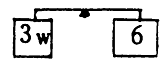


Therefore:

$$y + 3$$

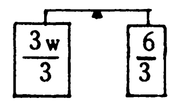


Given: $3w = 6$



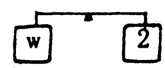
Divide both sides by 3 in order to get w alone.

$$\frac{3w}{3} = \frac{6}{3}$$

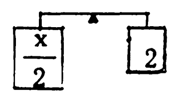


Therefore:

$$w = 2$$

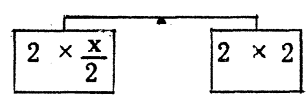


Given: $\frac{x}{2} = 2$



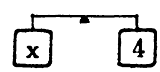
Multiply both sides by 2 in order to get x alone.

$$2 \times \frac{x}{2} = 2 \times 2$$



Therefore:

$$x = 4$$



The *terms* of an equation are the parts separated from each other by plus or minus signs or by the equal sign. Thus in $x + 4 = 7$, the terms are x, 4, and 7.

In $x + 4 = 7$, if each arithmetic number is replaced by a letter, the equation can be written

$$x + a = b$$

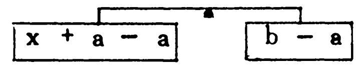
where $a = 4$ and $b = 7$. If both of the letters *a* and *b* are assigned any numerical values whatsoever, the numerical value of *x* can be found by the method just shown.

In an equation, where all the terms contain letters, one of the letters is assumed to be unknown (usually a letter near the end of the alphabet) and the other letters are assumed to be known. Such an equation can be solved for the unknown before arithmetic numbers are assigned to the letters. To do so, we follow the same procedure used above.

Given: $x + a = b$

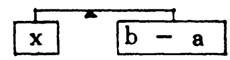
Subtract *a* from both sides.

$$x + a - a = b - a$$



Now $a - a = 0$ whether $a = 2$ or 3 or any other number.
Therefore:

$$x = b - a$$



In this equation, if arithmetic numbers are assigned to *a* and *b*, the particular value of *x* can be determined. If $a = 4$ and $b = 7$, then $x = 7 - 4 = 3$.

If $a = 6$ and $b = 3$, then $x = 3 - 6$. By methods used in arithmetic, *x* cannot be determined in this case because 6 cannot be subtracted from 3. This difficulty is overcome by introducing *negative numbers.*

$$x = 3 - 6$$

$$x = -3$$

Thus a new concept of algebra is introduced. Up to this point in our discussion, the signs + and − have been used to indicate the fundamental operations of addition and subtraction. The same signs, however, are used in algebra to indicate not only these two fundamental operations, but also the quality of a number, that is, whether the number is *positive* or *negative;* where positive and negative are opposite directions or opposite senses.

If going *East* is considered a positive direction, going *West* is a negative direction; or, if going West is considered a positive direction, going East is a negative direction.

If going *upstairs* is a positive direction, going *downstairs* is is a negative direction; or vice versa.

In business we have *credits* and *debits.* A debit can be thought of as a negative credit.

Room temperature is 68^0 above zero Fahrenheit, that is +68°F. Air is changed to a liquid at 300° below zero Fahrenheit, that is -300°F. Thus *above zero* is taken as positive and *below zero* is then negative.

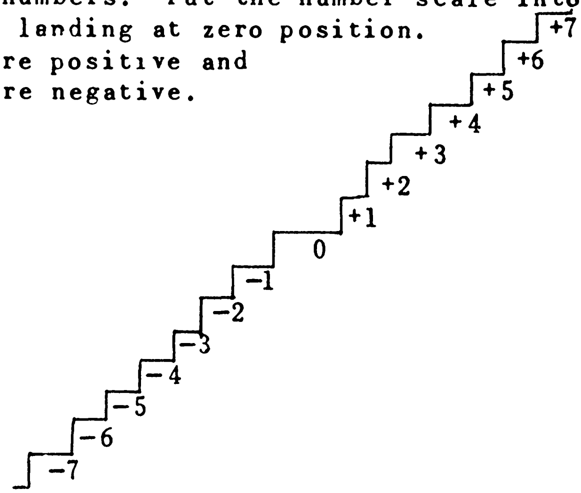
Increases and *decreases* in the closing prices of stocks are indicated by plus and minus signs respectively, +¾ means that the closing price is $¾ higher than that of the previous day; and -1¼ means that the closing price is $1¼ lower than that of the previous day.

Signed or directed numbers

The positive and negative numbers of algebra are called *signed* or *directed numbers.* They can be represented as the positions on a line, called the *algebraic number scale.* Zero is the dividing point. To the right are the one unit (centimeter, foot, mile or whatever unit is desired) positive signed numbers and to the left are the negative signed numbers. (All terms without a sign are taken as positive.)

-5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5

By using the number scale, the four fundamental operations can be performed with signed numbers. Put the number scale into the form of a stairway with a landing at zero position. Above the landing the steps are positive and below the landing the steps are negative.



Examples of addition of signed numbers

1) (+4) + (+3) = +7

From step +4, go *up* 3 steps; or from +4, count 3 units to the *right*.

2) (+4) + (−3) = +1

From step +4, go *down* 3 steps; or from +4, count 3 units to the *left*.

3) (−4) + (−3) = -7

From step −4, go *down* 3 steps; or from −4, count 3 units to the *left*.

Examples of subtraction of signed numbers

1) (+4) − (+3) = +1

Recall that subtraction is finding that number (the remainder) which added to the number to be subtracted (the subtrahend), +3 in this case, gives the number from which we are subtracting (the minuend), +4 in this case. In the above example this means finding what number added to +3 gives +4. Hence from step +3, go up (positive direction) 1 step to reach step +4; or from +3, count to the *right* (positive direction) 1 unit to reach +4.

(+4) − (−3) = +7

From step −3, go *up* 3 steps to reach the landing place and then go to step +4, making altogether 7 steps *up*; or from −3, count to the *right* 7 units to reach +4.

3) $(-4) - (+3) = -7$

From step +3, go *down* (negative direction) 3 steps to reach the landing place and then go down to step −4, making altogether 7 steps *down*; or from +3, count to the *left* (negative direction) 7 units to reach −4.

4) $(-4) - (-3) = -1$

From step −3, go *down* 1 step to reach step −4; or from −3, count to the *left* 1 unit to reach −4.

5) $0 - (+4) = -4$

From step +4, go *down* 4 steps to reach the landing place; or from +4, count to the *left* 4 units to reach zero.

6) $0 - (-4) = +4$

From step −4, go *up* 4 steps to reach the landing place; or from −4, count to the right 4 units to reach zero.

Examples of multiplication of signed numbers

Multiplication is a short process for adding. Thus

1) $(+3)(+4) = 3 \times (+4) = (+4) + (+4) + (+4) = +12$

2) $(+3)(-4) = 3 \times (-4) = (-4) + (-4) + (-4) = -12$

Since $2 \times 3 = 3 \times 2$, or $3 \times 4 = 4 \times 3$, then $ab = ba$. Therefore

3)
$$(-3)(+4) = (+4)(-3) = 4 \times (-3)$$
$$= (-3) + (-3) + (-3) + (-3) = -12$$

Or, since $(+3)(+4)$ can be thought of as adding +4 to zero three times; then $(-3)(+4)$ can be thought of as subtracting +4 from zero three times; or

$$(-3)\ (+4) = 0 - (+4) - (+4) - (+4) = -12$$

Since $-(+4) = -4$ [see 5) under *Examples of subtraction of signed numbers*], the problem can be written:

$$(-3)(+4) = -4 - 4 - 4 = -12$$

Now consider $(-3)(-4)$. It can be thought of as subtracting −4 from zero three times; or

4) $(-3)(-4) = 0 - (-4) - (-4) - (-4) = +12$

Since $-(-4) = +4$ [see 6) under *Examples of subtraction of signed numbers*], the problem can be written:

$$(-3)(-4) = +4 + 4 + 4 = +12$$

If a *friend* represents a *positive number* and an *enemy* represents a *negative number*, the following analogy can be used.

The friend of the friend is a friend.

1) $(+3)(+4) = +12$

The friend of the enemy is an enemy.

2) $(+3)(-4) = -12$

The enemy of the friend is an enemy.

3) $(-3)(+4) = -12$

The enemy of the enemy is a friend.

4) $(-3)(-4) = +12$

Examples of division of signed numbers

Division is the inverse of multiplication; therefore:

1) Since $(+3)(+4) = +12$, $\frac{+12}{+4} = +3$, or $\frac{+12}{+3} = +4$.

2) Since $(+3)(-4) = (-3)(+4) = -12$,

$\frac{-12}{-4} = +3$, or $\frac{-12}{+3} = -4$, or $\frac{-12}{-3} = +4$, or $\frac{-12}{+4} = -3$.

3) Since $(-3)(-4) = +12$, $\frac{+12}{-3} = -4$, or $\frac{+12}{-4} = -3$.

The *friend-enemy analogy* holds true for division as for multiplication.

Rules for the multiplication and division of signed numbers may be stated as follows:

1. *If both have the same sign, the product or quotient is positive.*

Illustrations:

a. $(+3)(+4) = +12$
b. $(-3)(-4) = +12$
c. $\frac{+12}{+4} = +3$
d. $\frac{-12}{-4} = +3$

2. *If one number is positive and the other is negative, the product or quotient is negative.*

Illustrations:

a. $(+3)(-4) = -12$
b. $(-3)(+4) = -12$
c. $\frac{+12}{-4} = -3$
d. $\frac{-12}{+4} = -3$

In practice, problems involving signed numbers are written with a minimum number of signs and parentheses. $+ (+4)$ is written $+4$. In like manner $-(-4) = +4$; $-(+4) = -4$; and $+(-4) = -4$. That is, like signs make a positive sign; and unlike signs make a negative sign.

Now consider the following problem which uses some of these rules of signs.

If $2x + 9 = 6$, what is the value of x?

Given: $2x + 9 = 6$

Subtract 9 from both sides in order to get x alone.

$$2x + 9 - 9 = 6 - 9$$

Since $6 - 9 = -3$ (See Examples of subtraction of signed numbers.),

$$2x = -3$$

Divide both sides by 2 in order to get x alone.

$$\frac{2x}{2} = \frac{-3}{2}$$

Or,

$$x = -1\tfrac{1}{2}$$

Some equations contain parentheses. (See *Multiplication of a sum* on page 52.) To find the unknown of equations containing parentheses, the parentheses must be removed first. For example, if $2(x - 4) - 3(x + 1) = 5$, what is the value of x?

Given: $2(x - 4) - 3(x + 1) = 5$

Since $2(x - 4) = 2x - 2 \times 4 = 2x - 8$ and $-3(x + 1) = -3x - 3 \times 1 = -3x - 3 \times 1 = -3x - 3$, the equation can be written

$$2x - 8 - 3x - 3 = 5$$

We combine like terms as follows: $2x - 3x = -1x$ or $-x$ and $-8 - 3 = -11$. Hence

$$-x - 11 = 5$$

Next add 11 to both sides, then

$$-x - 11 + 11 = 5 + 11$$

Or

$$-x = 16$$

Since $-x$ means $(-1) \times x$, we divide both sides by -1 that is,

$$\frac{(-1) \times x}{(-1)} = \frac{16}{-1}$$

From which $x = -16$ (See l.d. p. 61)

Equations with two unknowns

Many problems exist that involve two unknowns. Consider the following simple illustrations.

1. I have a shelf just long enough to hold six five-pound packages of sugar and flour. If I have six sacks of sugar, I have no room for flour; if I have five sacks of sugar, I have room for one sack of flour; and so on. That is:

If I have the following number of sacks of sugar,	6	5	4	3	2	1	0
I have room for the following number of sacks of flour	0	1	2	3	4	5	6

Thus the number of sacks of sugar plus the number of sacks of flour equals six.

2. I wish to purchase some five-pound sacks of sugar and flour with the understanding that there will be two more sacks of sugar than sacks of flour. That is if I purchase ten sacks of sugar, I will also purchase eight sacks of flour; and so on. The following comparisons show some of the possible combinations.

If I purchase the following number of sacks of sugar,	10	8	7	6	5	4	2
I must purchase the following number of sacks of flour	8	6	5	4	3	2	0

Thus the number of sacks of sugar purchased minus the number of sacks of flour purchased is two.

3. If now I wish to satisfy at the same time *both* conditions set up in illustrations 1. and 2., on a shelf that holds only six sacks of sugar and flour I want to place two more sacks of sugar than flour. The possible combinations given above, show that the only possible combination is to purchase *four* sacks of sugar and *two* sacks of flour.

Algebraically we can say this in the following elegent, condensed form.

If x = the number of sacks of sugar and if y = the number of sacks of flour, in the first illustration

$$x + y = 6,$$

in the second illustration

$$x - y = 2,$$

and, in the third illustration the two equations

$$\begin{cases} x + y = 6 \\ x - y = 2. \end{cases}$$

are considered at the same time (simultaneously); and as we see from our sets of values above $x = 4$ and $y = 2$.

Suppose now we met the more general problem which gives the same equations: The sum of two numbers is 6 and the difference of the same two numbers is 2. What are the numbers? To find them, suppose we let x = the larger number and y = the smaller number. Then

$x + y = 6$ (their sum is 6)

$x - y = 2$ (their difference is 2)

For $x + y = 6$, if any arithmetic number whatsoever is assigned to either x or y, the value of the other letter can be determined. Each pair of values determined satisfy the equation; that is, the left side of the equality equals the right side. Thus, if $x = 4$, $4 + y = 6$, or $y = 2$. Other pairs of values are shown below; and, of course we could use the values used in the sugar and flour problem.

x	0	1	2	3	4	5	6	7	8
y	6	5	4	3	2	1	0	-1	-2

and so on.

Similarly for $x - y = 2$, pairs of values of x and y that will satisfy this equation are:

x	0	1	2	3	4	5	6	7	8
y	-2	-1	0	1	2	3	4	5	6

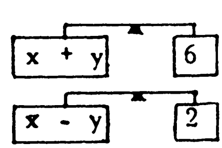
and so on.

Considering the equations $x + y = 6$ and $x - y = 2$ at the same time, that is, considering the series of possible values for each equation, shown above, one sees that both sets contain $x = 4$ and $y = 2$ and no other common set. These values are said to satisfy both equations $x + y = 6$ and $x - y = 2$. This common pair of values ($x = 4$, $y = 2$) is called the solution, meaning the common solution, of the *simultaneous equations*, $x + y = 6$ and $x - y = 2$.

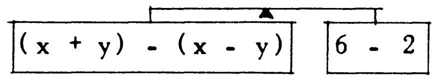
The above trial method is too tedious, so we use shorter methods as we shall now illustrate.

Given: $x + y = 6$ 1)

$x - y = 2$ 2)



Subtract the second equation from the first. In doing so, use parentheses. We obtain



$$(x + y) - (x - y) = 6 - 2$$

Since - (x - y) means (-1) x (x - y) or (-1) x x - (-1) x y or -x + y, this equation can be written in the form

$$x + y - x + y = 4$$

Combining like terms, we have

$$2y = 4$$

Whence

$$y = 2$$

We now substitute this value for y in equation 1). This gives

$$x + 2 = 6$$

Whence

$$x = 4$$

Therefore x = 4 and y = 2 are the values of the two unknowns that satisfy *both* equations.

Graphical solution of simultaneous equations

Suppose we are going to see friends in the City of San Diego. We are told by these friends that it is very easy to locate their homes since the town is ideally layed out. (See Figure 1.) The *North-South* streets *East* of Front St. are named 1st Ave., 2nd Ave., and so on. The *East-West* streets *South* of Ash St. are named A St., B St., and so on; and the *East-West* streets North of *A*sh St. are *B*eech St., *C*edar St., and so on.

By going to an intersection, such as *Broadway* and *Front*, it is easy to find homes of friends. Suppose we wanted to find such street intersections as those marked P, Q, R, and S. P is 7 blocks *East* and 6 blocks *North* of our starting point. If *East* and *North are* considered as *positive* directions, the intersection P can be designated as (7,6), that is *(East* 7 blocks to 7th Ave., *North* 6 blocks to Cedar St. - C St., B St., A St., Ash St., Beech St., Cedar St.).

To reach Q, go 2 blocks *West* and 6 blocks *North* of our starting point (*Broadway* and *Front*). Since *East* is assumed to be a *positive* direction, *West* is a *negative* direction. Therefore Q can be designated as (-2,6).

To reach R, go 3 blocks *West* and 1 block *South* from our starting point (*Broadway* and *Front*). Since *North* is assumed to be a *positive* direction, *South* is a *negative* direction. Therefore R can be designated as (-3,-1).

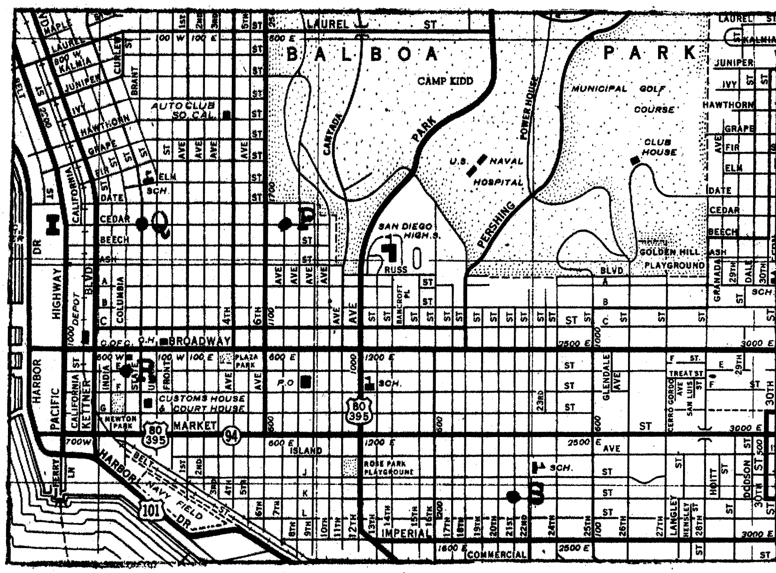


Figure 1

Finally, S can be designated as (21, -7).

Most towns use essentially the same plan. But all are more or less complicated by the naming of their streets. In algebra we use a similar diagrammatic scheme but much simpler yet filling the same need.

We lay out two algebraic number scales (See p. 59.) perpendicular to each other with the zero points of each at the same position, as in Figure 2.

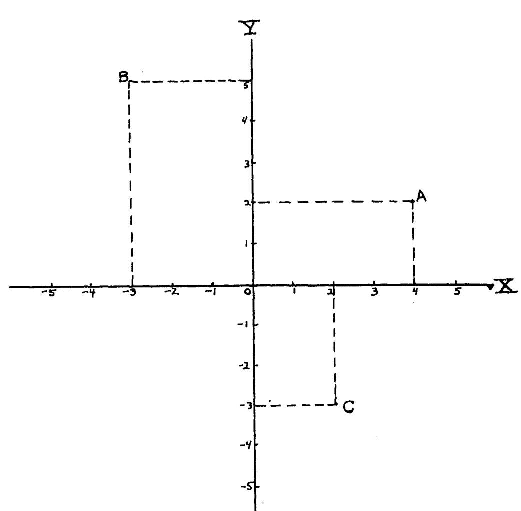


Figure 2

The horizontal line, called the *horizontal axis*, is used for the values of x. The vertical line, called the *vertical axis*, is used for the values of y. The horizontal axis is also called the *X-axis*, and the vertical the *Y-axis*. The point where the two axes cross is called the *origin*.

For point A, the x value is 4 and the y value is 2. A is spoken of as the point (4,2).

If several pairs of values that satisfy an equation in two unknowns are located on a diagram, such as Figure 2, and if these points are joined by a smooth line, this line is called *the graph of the equation*.

Illustration: To draw the graph of the equation $x + y = 6$, we determine several pairs of values that satisfy the equation by choosing convenient values for x substituting each value in the equation and solving for the corresponding value of y. Suppose we have thus obtained the following *pairs of values*.

x	0	1	2	3	4	5	6	7	8
y	6	5	4	3	2	1	0	-1	-2

and so on.

We then locate each *pair of values* on a diagram, as in Figure 3, and join them by a smooth line.

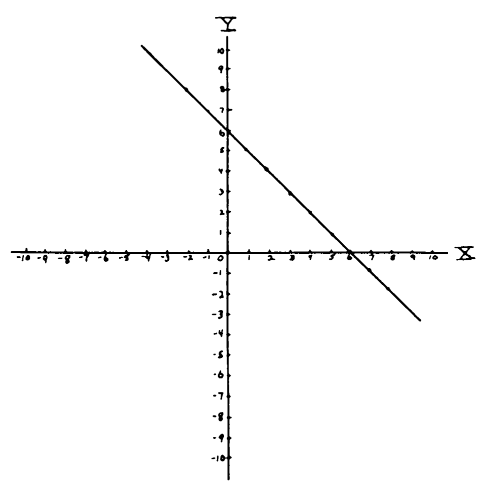


Figure 3

Any point selected on this line will have a definite x value and a definite y value. The values of x and y for every such point satisfy the equation $x + y = 6$. For example the point $(4\frac{1}{2}, 1\frac{1}{2})$ satisfies the equation since $4\frac{1}{2} + 1\frac{1}{2} = 6$.

Graphical solution of simultaneous equations

If the *graphs of two equations in x and y* are drawn on the same diagram, *the point* where the two lines intersect is *the solution of the two equations,* as we shall now show.

Any point selected on either line will have a definite x value and a definite y value. The values of x and y for each point on the line (1) satisfy the equation $x + y = 6$. The values of x and y for each point on the line (2) satisfy the equation $x - y = 2$. (See Figure 4.) The values of x and y for the *point of intersection* of the lines (1) and (2), namely $x = 4$ and $y = 2$, *satisfy both equations*. This is true without any use of algebra since two straight lines in the same plane always have just one point in common unless they are parallel.

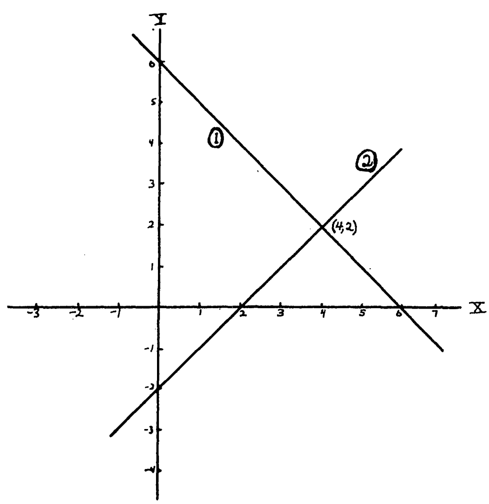


Figure 4

It is the author's sincere hope that through this article the reader will become interested in a further pursuance of the subject. A knowledge of algebra and equations is essential for an understanding of modern science. Algebra is the basis of all mathematics, and on mathematics depends engineering, physics, chemistry, and in fact all scientific work.

Los Angeles City College

MATHEMATICS MAGAZINE

Edited by

C. G. Jaeger and H. J. Hamilton

This department will submit to its readers, for solution, problems which seem to be new, and subject-matter questions of all sorts for readers to answer or discuss, questions that may arise in study, research or in extra-academic applications.

Contributions will be published with or without the proposer's signature, according to the author's instructions.

Although no solutions or answers will normally be published with the offerings, they should be sent to the editors when known.

Send all proposals for this department to the Department of Mathematics, Pomona College, Claremont, California.

SOLUTIONS

No. 3. Proposed by Nev. R. Mind.

If the medians of a triangle are proportional to the corresponding sides, the triangle is equilateral.

Solution by C. W. Trigg, Los Angeles City College.

It is well-known that $m_a = \frac{1}{2}\sqrt{2(b^2+c^2)-a^2}$. By the hypothesis, $\dfrac{\frac{1}{2}\sqrt{2(b^2+c^2)-a^2}}{a} = \dfrac{\frac{1}{2}\sqrt{2(c^2+a^2)-b^2}}{b}$.

Upon simplification, this equation takes the form

$$(a - b)(a + b)(a^2 + b^2 + c^2) = 0.$$

Since the last two factors cannot equal zero, $a = b$. Proceeding in like manner with the ratio $m_b : b :: m_c : c$, we have $b = c$. Therefore the triangle is equilateral.

Also solved by Barney Bissinger, Fitchburg, Mass.

No. 4. Proposed by Pedro A. Piza, San Juan, Puerto Rico.

Let the integers a, b, c, with $c = a + 1$, be the sides of a right triangle. Show that $b^2c^2 + a^4 = a^2c^2 + b^4$ and that this value increased by 3 is a perfect square.

Solution: Since $b^2a^2 = a^2b^2$ and $a^2 + b^2 = c^2$, we have

$$b^2(c^2 - b^2) = a^2(c^2 - a^2) \text{ or } b^2c^2 + a^4 = a^2c^2 + b^4$$

which is independent of the condition $c = a + 1$

Using $c = a + 1$ in $a^2c^2 + b^4 + 3$ we have

$$a^2(a + 1)^2 + (c^2 - a^2)^2 + 3 = a^2(a + 1)^2 + \left\{(a + 1)^2 - a^2\right\}^2 + 3$$

which when simplified becomes

$$a^4 + 2a^3 + 5a^2 + 4a + 4 = (a^2 + a + 2)^2$$

Barney Bissinger

Fitchburg, Mass.

Also solved by C. W. Trigg, Los Angeles, California; Harold Bowie, Indian Orchard, Mass.; Thomas Griselle, Hollywood, Calif., and Lawerence A. Ringenberg, Charleston, Illinois who gave the parametric equations

$$a^2 = 2\nu^2 + 2\nu, \quad b = 2\nu + 1, \quad c = 2\nu^2 + 2\nu + 1$$

for triangles of this type.

No. 5. Proposed by Victor Thebault, Sarthe, France.

Using once each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, form a number which when increased by one million becomes a perfect square.

Solution by C. W. Trigg, Los Angeles City College.

If $N + 10^6 = M^2$ then $N = (M + 10^3)(M - 10^3)$. Any number is congruent to the sum of its digits modulo 9, so $N \equiv 45 \equiv 0 \pmod 9$. Hence $M \equiv \pm 1 \pmod 9$. Now $1024456789 \leq M^2 \leq 9877543210$, so $32007 \leq M \leq 99386$. Only those values of M within this range, which meet the congruence requirement, need be examined. The search may be aided by using a table of squares of four-digit integers and an auxiliary table of squares with four distinct terminal digits. Thus many solutions are found. For example:

$M \equiv 1$	N	$M \equiv -1$	N
35902	1287953604	40445	1634798025
38836	1507234896	50741	2573649081
45433	2063157489	53801	2893547601
52966	2804397156	54584	2978413056
53407	2851307649	56555	3197468025

No. 6. Proposed by Fred Fender, South Orange, New Jersey

Solution by Raymond H. Wilson Jr., Temple University, Philadelphia

Assuming fuel weight = 3W/2, where W is the empty weight of the rocket, the mass to be accelerated at any time, *t* seconds from the takeoff, would be

$$\frac{W}{g}\left[1+\frac{3}{2}\left(\frac{100-t}{100}\right)\right] = \frac{W}{g}\left(\frac{500-3t}{200}\right) \qquad t \leq 100$$

where $g = 9.8$ meters/sec^2. Also, from the theory of jet propulsion, the reaction force of the jet for any velocity, v meters/-sec., of the rocket would be

$$\frac{3W/2}{100}(4900-v) = \frac{3W}{200}(4900-v).$$

Hence, the Newtonian equation of motion of the rocket for $t \leq 100$ would be

$$\frac{3W}{200}(4900-v) - \frac{W}{200}(500-3t) = \frac{W}{200g}(500-3t)\frac{dv}{dt},$$

or

$$\frac{dv}{dt} + \left(\frac{3g}{500-3t}\right)v = g\left[\frac{3(4900)}{500-3t}-1\right].$$

Integrating this as a standard linear differential equation of the first order, and applying the condition $v = 0$ when $t = 0$,

$$v = 4900\left[1-\left(\frac{500-3t}{500}\right)^{g}\right] - \frac{g(500-3t)}{3(g-1)}\left[1-\left(\frac{500-3t}{500}\right)^{g-1}\right] = \frac{dz}{dt},$$

where z = height above starting point on the earth's surface. Direct integration, applying the condition $z = 0$ when $t = 0$, gives

$$z = 4900t - \frac{g(500)^2}{18(g-1)}\left[1-\left(\frac{500-3t}{500}\right)^{2}\right] + \left[\frac{2g(500)^2}{18(g-1)(g+1)} - \frac{(4900)(500)}{3(g+1)}\right]\left[1-\left(\frac{500-3t}{500}\right)^{g+1}\right]$$

After $t = 100$, when the fuel is burned out, the rocket will continue as a freely body, and we have the familiar equation for maximum height, Z, of a projectile:

$$Z = z_{100} + \frac{(v_{100})^2}{2g}.$$

By logarithmic computation may be obtained:

$$z_{100} = 404,259 \text{ meters} \qquad v_{100} = 4825.2 \text{ meters/sec.},$$

and, finally,

$$\boxed{Z = 1592.2 \text{ kilometers.}}$$

If the fuel weight is assumed to be only one-third as great, the maximum height attained is 1404.4 kilometers.

In reality g is a function of z which could be included in the equation of motion, but then this equation would be no longer simple and probably impossible of solution in finite form. A fairly good approximation might be made by assuming $g = 9\text{m./s}^2$ for the first 100 seconds; and $g = 7\text{m./s}^2$ for the free flight motion after this to the peak. A rough estimate indicates a resultant increase in maximum height of some 200 kilometers.

Air resistance up to 100 km. or so would be very considerable. If proper constants were known, it might be expressed as a function of velocity and height (upon which pressure depends), and thus be included in the equation of motion, which would thereby become still more complicated. This resistance would decrease the maximum height, so the stated assumptions of the problem tend to be self-compensating.

MATHEMATICAL MISCELLANY

Edited by

Marian E. Stark

Let us know (briefly) of unusual and successful programs put on by your Mathematics Club, of new uses of mathematics, of famous problems solved, and so on. Brief letters concerning the Mathematics Magazine or concerning other "matters mathematical" will be welcome. Address: Marian E. Stark, Wellesley College, Wellesley 81, Mass.

The following information concerning mathematics and mathematicians in France comes to us in a letter received by Colonel Byrne of our editorial staff from Professor Gaston Julia, who has the chair of Analyse Supérieure at the Sorbonne and also the chair of Géométrie at the École Polytechnique. The letter was written on November 2, 1947. Professor Émile Borel retired in 1942 and Professor Paul Montel in October 1946. Professor Julia has not yet received the poster showing the courses offered for the first semester at the Sorbonne. The journals being published at present are:

Annales de l'École Normale Superieure
Journal de Mathématiques Pures et Appliquees
Bulletin de la Société Mathématique de France
Bulletin des Sciences Mathématiques

These journals are quite behind schedule ("La publication a de *gros retards.*") The Annales de l'Institut Poincaré have not resumed publication. Paper is lacking and electric power for the presses is not always available. Printing delays for memoirs and books are very long (sometimes 2 or 3 years). Professor Julia tells us that he spent the month of September, 1947, lecturing in Stockholm and Oslo. Colonel Byrne adds that he knows that the "Intermédiaire des Recherches Mathématiques" has been published since 1945. The eleventh number (July 1947) is the latest (as of November 10, 1947). The elementary periodicals (mostly devoted to examination questions, including the competitive Agrégation examinations for teachers) Revue de Mathématiques Spéciales and Journal de Mathématiques Élémentaires, published by the Librairie Vuibert (also L'Éducation Mathématique), appear regularly now.

In the book "Entertainments in the Little Theatres of Madrid, 1759-1819" by Ada M. Coe we find the following:

"Mathematical Tricks"
(Juegos de Matematica)
1768

For unadorned 'geometry' D. Faustine de Muscat y Guzman could charge one *real* admission fee; would demonstrate the reduction of a circle to a square. Doubters could come and see it done, then the following day go to the local bookstore and buy the book whose diagrams would reveal the secret. Under the unexpected caption of 'mathematics' sleight of hand tricks became *juegos de manos de matematica y fisica,* and when sponsored by Daniel Peragallo, pupil of the 'celebrated Pinetti,' the repertory included tricks of physics, mathematics, arithmetic, playing cards."

Can any of our readers supply information about similar mathematical "entertainments" at about that same time in other lands?

Hidden Mathematicians

1. He gave me his card and left.
2. Frank accepted the offer; Matthew refused it.
3. He heard the bell tinkle in the hall.
4. His house was new; to name it was the next problem.
5. It was a Mercedes car, Tess.
6. There was I with a broken arch, I. Medes and Persians how I suffered!
7. Cain slew his brother Abel.
8. I hate cant or blah-blah.
9. The band was uniformed in white, headed by a drum-major in red.
10. On her lap lace of the finest design lay.
11. What could I do but love her, mite that she was?
12. Agnes, if I go, will you stay at home?
13. Ah, mes amis. bonjour!
14. If he is an anti phone the police at once.
15. I say Edna, pie reveals your culinary talent.
16. Je ne veux pas de lucre, mon ami.
17. I gave him back four, i.e. returned all but one.
18. He told them with simple charm the legend regarding the headless rider.
19. It was as fine a tulip as California could produce.
20. His horse would always lag, range the hills as we might.
21. For example, take plerome. Only a botanist knows what it is.
22. After studying at a technical school he became interested in ore smelting.
23. They say that his garden wall is something to behold.
24. According to his theory there is no ether necessary for the explanation of wave motion.
25. The dolphin appears on the surface frequently.

Boston University

Elmer B. Mode.

OUR CONTRIBUTORS

Marshall H. Stone, Professor of Mathematics, University of Chicago, was born in New York City in 1903. His father was the late Chief Justice Harlan F. Stone of the U.S. Supreme Court. Educated at Harvard (A.B. '22, A.M. '24, Ph.D. '26), Prof. Stone served on the faculties of Columbia, Harvard, and Yale, and then returned to Harvard where he was appointed professor in 1937 and elected chairman of the department of mathematics at the University of Chicago.

A member of the National Academy of Sciences and the Philosophical Society, etc., he has taken a leading part in the work of the American Mathematical Society and has served as President of the Society and member of the Council. Widely known for his researches in general analysis and Boolean algebras, and for his colloquium volume, "Linear Transformations in Hilbert Space," Prof. Stone has been an assoc. editor of the Transactions, the Annals, and the American Journal.

Harry S. Vandiver, Professor of Mathematics, University of Texas, was born in Philadelphia in 1882, and attended the University of Pennsylvania in 1904-05. From 1900-17, Prof. Vandiver was associated with his father as customs house broker in Philadelphia. During the first world war he served with the U.S.N.R.F., and after returning from the war he was appointed instructor at Cornell. Prof. Vandiver joined the faculty at Texas as an associate professor ('24), became professor ('35), and distinguished professor of mathematics (Dept. Appl. Math., '47)., Prof. Vandiver has received various honors for his extensive researches in algebra and number theory, including the Cole Prize ('31) and election to the Nat. Acad. of Sciences. Prominent in the Am. Math. Soc., he was named colloquium lecturer ('35) and Vice President ('35-'37). He has also served as chairman of the committee on algebraic numbers, National Research Council (1923-28) and as assistant editor of the Annals (1926-39).

René Maurice Fréchet, Prof., Inst. Poincaré, University of Paris, and Prof., Ecole Norm. Sup., Paris and St. Cloud, was born in Maligny, France, in 1878. After receiving the doctorate at Paris ('06), he was appointed successively Maitre des Conferences, Rennes ('08), Prof. de mecanique, Poitiers ('10), Prof., Analyse Sup. and Dir., Inst. Math., Fac. de Sci., Strasbourg ('19). At Strasbourg he also was named Prof., Assur. et Statist., Inst. d'Enseignment commerc. sup. Prof. Fréchet has held his present position since 1928.

Internationally famous among mathematicians both for his pioneering work on abstract spaces and his many researches in analysis, probability and statistics, Prof. Fréchet is the author of a number of books on various advanced phases of mathematics, perhaps the best known being "Les Espaces Abstracts." He recently returned to France after an extended lecture tour at American colleges and universities. An article by him explaining his recent theory of abstract random elements will appear in a forthcoming issue of this magazine.

E. J. Hills was born in Winter Park, Florida, in 1899, and attended Indiana University (A.B. '21), California (M.A. '26), Harvard and Southern California (Ph.D. '41). Since 1930 Dr. Hills has been on the Faculty of the Los Angeles City College. A specialist in mathematical education, Dr. Hills is a member of the Mathematics Education Committee for California of the Mathematical Association and is the author or coauthor of three recent texts at the junior college level.